SINGULAR POINTS OF ANALYTIC TRANSFORMATIONS*

BY

WILLIAM F. OSGOOD

The object of the present paper is to discuss the nature of an analytic transformation in the neighborhood of a singular point. Let

$$(A) \qquad x_i = \phi_i(u_1, \cdots, u_n) = \frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)} \qquad (i = 1, \cdots, n),$$

where the functions $\phi_i(u_1, \cdots, u_n)$ are meromorphic in the origin, $(u) = (0)$, and at least one of them has a non-essential singularity of the second kind there.

The functions g_i, G_i are all analytic in the origin, and two of these functions corresponding to the same value of i cannot, of course, both vanish identically. We interpret the point (x) in the space of analysis, and thus it is not necessary to exclude the case that a function G_i vanishes identically.

When two functions g_i, G_i both vanish in the origin, it shall be assumed that they have no common factor there.† In particular, then, if a function g_i or G_i vanishes identically, the other function shall be taken as not vanishing, and may be set equal to unity.

In all cases, at least one pair of functions, as g_i and G_i, vanishes in the origin, and the transformation breaks down at this point (and at others, too, in the neighborhood, when $n > 2$).

Let the positive numbers η_i $(i = 1, \cdots, n)$ be chosen arbitrarily small, and certainly small enough so that each function g_i, G_i is analytic throughout the region

$$|u_i| < \eta_i \qquad (i = 1, \cdots, n).$$

Let the points at which two functions g_i, G_i vanish simultaneously be excluded from this region, and let the remaining region be denoted by $\mathfrak{T}$. Then each point (u) of $\mathfrak{T}$ is carried over into a single point of the (x)-space, and these latter points form a certain manifold, M.

As the quantities η_i are taken smaller and smaller and $\mathfrak{T}$ is thus reduced

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† The leading theorems relating to the factorization of functions of several complex variables which are analytic in a point and vanish there are contained in a memoir by Weierstrass, *Werke*, vol. 2, p. 135. For a systematic presentation of the theory cf. a paper by the author in the Annals of Mathematics, ser. 2, vol. 19 (1917), p. 77.

in extent, this manifold M continually loses points. But there are certain points of the (x)-space which are limiting points of M, no matter how small $\mathfrak{T}$ be taken. *The set of these points shall be denoted by* $\mathfrak{M}$.

It is the study of this latter manifold to which this paper is devoted, and a complete solution of the problem is obtained.*

The result of the investigation is embodied in the theorem of § 10. But the reader will find it convenient to begin with the special cases $\mu = 1, 2, 3$ studied in §§ 1, 6, and 8, respectively.

A further problem, which is in a way a continuation of the present one, is the following. Let

$$x_i = \phi_i(u_1, \cdots, u_n) \qquad (i = 1, \cdots, n),$$

where ϕ_i is meromorphic throughout a fixed region, S, of the (u)-space. Let σ be the manifold of points in S, in which at least one ϕ_i has a non-essential singularity of the second kind, and let σ contain at least one point. Imbed σ in a region T of S, remove the points of σ from T, and denote the remainder of T by $\mathfrak{T}$. Then each point of $\mathfrak{T}$ is carried over by the above transformation into a point of the (x)-space. Denote the manifold of the latter points by N.

As $\mathfrak{T}$ is steadily reduced in extent, so that an arbitrary point of S not lying in σ ultimately becomes and remains an exterior point of $\mathfrak{T}$, there are certain points of the (x)-space which always persist as limiting points of N. *The totality of these latter points forms a manifold* $\mathfrak{N}$.

In the simplest case, namely, $n = 2$, σ consists of isolated points, and hence, if S be taken as closed, $\mathfrak{N}$ consists of a finite number of manifolds $\mathfrak{M}$, each of which is made up of a finite number of algebraic plane curves.

The author reserves the further study of this problem for a later occasion.

1. The Case $\mu = 1$. Preparation for the Higher Cases

Let the equations (A) be so arranged that the first μ functions

$$\frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)} \qquad (i = 1, \cdots, \mu),$$

have a non-essential singularity of the second kind in the origin, the remaining functions, $i = \mu + 1, \cdots, n$, either being analytic there or having at most a pole. Then $1 \leqq \mu \leqq n$.

The Case $\mu = 1$. This case is immediately disposed of, since x_1 can actually assume and retain any preassigned value, as the point (u) approaches the

* This problem has been studied by Autonne, Acta Mathematica, vol. 21 (1897), p. 249, and some of the results of the present paper are there given. But the treatment is rather a sketch than a detailed investigation. It is not possible to read between the lines that which is necessary for rigorous proofs.

origin, $(u) = (0)$, along a suitable path. No matter what the mode of approach, each x_i, $1 < i$, approaches a definite limit, $x_i = a_i$. Hence $\mathfrak{M}$ is seen to consist of the right line

$$|x_1| \leqq \infty, \qquad x_i = a_i \qquad (i = 2, \cdots, n).$$

The First of Equations (B), and the Regions R_1, Σ_1, $\bar{R}_1$, etc. When $\mu > 1$, we set over against equations (A) the following equations:

$$(B) \qquad \begin{cases} g_1(u_1, \cdots, u_n) - x_1 G_1(u_1, \cdots, u_n) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n(u_1, \cdots, u_n) - x_n G_n(u_1, \cdots, u_n) = 0. \end{cases}$$

Every solution of (A), where (x) is a point of the space of analysis, is a solution of (B). But a solution of (B) yields a solution of (A) only when, for no value of i, do g_i and G_i vanish simultaneously.

We will begin with the first of the equations (B),

$$(1) \qquad g_1(u_1, \cdots, u_n) - x_1 G_1(u_1, \cdots, u_n) = 0.$$

We may assume without loss of generality that

$$g_1(0, \cdots, 0, u_n) \not\equiv 0, \qquad G_1(0, \cdots, 0, u_n) \not\equiv 0,$$

since by means of a suitable linear transformation of the variables $u_1, \cdots, u_n$ these conditions can always be fulfilled.

If, furthermore, Au_n^p and Bu_n^q are the terms of lowest dimension in u_n when the functions $g_1(0, \cdots, 0, u_n)$ and $G_1(0, \cdots, 0, u_n)$ respectively are expanded into power-series in u_n, then we may assume that $p = q$, since a suitable linear transformation of x_1,

$$x_1' = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta},$$

together with the transformation

$$g_1'(u) = \alpha\, g_1(u) + \beta\, G_1(u),$$
$$G_1'(u) = \gamma\, g_1(u) + \delta\, G_1(u)$$

will replace (1) by a new equation of the desired type, while on the other hand belonging to the group of the space of analysis.*

The term of lowest dimension in u_n when the function

$$g_1(0, \cdots, 0, u_n) - x_1 G_1(0, \cdots, 0, u_n)$$

is expanded according to powers of u_n is, then,

$$(A - Bx_1)u_n^p, \qquad A \not\equiv 0, \qquad B \not\equiv 0.$$

* This transformation is made for convenience of presentation. Without it, the investigation that follows can be carried through with only formal changes.

The hyperplane obtained by putting the coefficient of u_n^p equal to 0 is the manifold

$\mathfrak{S}_1$: $$A - Bx_1 = 0.$$

From the closed (x)-space we remove the points of $\mathfrak{S}_1$ and denote the remaining space by R_1,—

R_1: $$0 < |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n).$$

We shall sometimes have occasion to consider only the space of the variables $(x_1, \cdots, x_k)$, $k < n$, and then, only that part of this space for which these variables satisfy the above relations. This region we shall denote by R_1^k,

R_1^k: $$0 < |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k).$$

Finally, let a neighborhood Σ_1 of $\mathfrak{S}_1$,

Σ_1: $$|A - Bx_1| < h, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n),$$

be removed from the (x)-space, and let the remaining space, which is closed, be denoted by $\bar{R}_1$,

$\bar{R}_1$: $$h \leqq |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, n).$$

The regions

Σ_1^k: $$|A - Bx_1| < h, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k);$$

$\bar{R}_1^k$: $$h \leqq |A - Bx_1| \leqq \infty, \qquad |x_j| \leqq \infty \qquad (j = 2, \cdots, k),$$

are now self-explanatory.

Solution of Equation (1). Let $(\xi) = (\xi_1, \cdots, \xi_n)$ be any point of $\bar{R}_1$, and restrict (ξ) to begin with to the finite region. Then the roots of (1) which lie in the neighborhood of the point $(u_1, \cdots, u_n, x_1) = (0, \cdots, 0, \xi_1)$ will be given by the equation*

(2) $$u_n^p + A_1 u_n^{p-1} + \cdots + A_p = 0,$$

where

$$A_i = A_i(u_1, \cdots, u_{n-1}, x_1)$$

is analytic in the point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1)$ and vanishes there.

Next, if ξ_1' is any point of a certain neighborhood of ξ_1 and the equation (2) is written down for the new point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1')$, the new coefficients A_i' will coincide respectively throughout a certain neighborhood of the point $(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \xi_1')$ with the coefficients A_i, considered for the same neighborhood.

From this it is seen that the coefficients A_i admit analytic continuation along every finite path of the manifold whose points $(u_1, \cdots, u_{n-1}, x_1)$ are

* Weierstrass, *Werke*, vol. 2, p. 135. Cf. also *Madison Colloquium*, pp. 181–183.

subject to the condition

$$u_k = 0, \qquad k = 1, \cdots, n-1; \qquad h \leqq |A - Bx_1| < \infty .$$

They can, however, also be continued analytically to the point

$$(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, \infty)$$

along a path in said manifold. For, let G be taken so that

$$G > \left|\frac{A}{B}\right|,$$

and let

$$x_1' = \frac{1}{x_1}.$$

Then, if $|x_1| \geqq G$ and if (1) is replaced by

$$x_1' g_1(u_1, \cdots, u_n) - G_1(u_1, \cdots, u_n) = 0, \tag{1'}$$

the roots of (1') which lie in a certain neighborhood of the point $(0, \cdots, 0, \xi_1')$, where

$$|\xi_1'| \leqq \frac{1}{G},$$

will be given by an equation

$$u_n^p + A_1' u_n^{p-1} + \cdots + A_p' = 0, \tag{2'}$$

where

$$A_i' = A_i'(u_1, \cdots, u_{n-1}, x_1')$$

is analytic in the point $(u_1, \cdots, u_{n-1}, x_1') = (0, \cdots, 0, \xi_1')$ and vanishes there.

These coefficients A_i' can be continued analytically along any path of the manifold

$$u_k = 0, \qquad k = 1, \cdots, n-1; \qquad |x_i'| \leqq \frac{1}{G}.$$

In the neighborhood of any one of these points $(0, \cdots, 0, \xi_1')$ for which $\xi_1' \neq 0$,

$$A_i'(u_1, \cdots, u_{n-1}, x_1') = A_i(u_1, \cdots, u_{n-1}, x_1),$$

and hence the above statement is seen to be true.

Finally, since h can be taken arbitrarily small, we are led to the following result. *The coefficients* $A_i(u_1, \cdots, u_{n-1}, x_1)$, $i = 1, \cdots, p$, *in* (2) *are single-valued and analytic at every point of the manifold*

$$(u_1, \cdots, u_{n-1}, x_1) = (0, \cdots, 0, x_1), \qquad 0 < |A - Bx_1| \leqq \infty, \tag{3}$$

and vanish there.

Concerning the algebroid polynomial F that forms the left-hand side of (2) we can assert furthermore that it is *irreducible, i. e.,* that a relation of the form cannot exist:

$$F = (u_n^q + B_1 u_n^{q-1} + \cdots)(u_n^s + C_1 u_n^{s-1} + \cdots),$$

where the coefficients B_j and C_k are functions of $(u_1, \cdots, u_{n-1}, x_1)$ analytic at all points $(0, \cdots, 0, \xi_1)$, ξ_1 being any point of R_1^1. For,

(*a*) F cannot have two essentially distinct factors. Let ξ_1 be any point of R_1^1. Then it is clear that a point $(u_1', \cdots, u_{n-1}')$ can be chosen arbitrarily near the origin and a point ξ_1' arbitrarily near ξ_1 so that, if $u_n^{(1)}$ and $u_n^{(2)}$ denote any two roots of (2) corresponding to the point $(u_1, \cdots, u_{n-1}, x_1) = (u_1', \cdots, u_{n-1}', \xi_1')$, then

$$G_1(u_1', \cdots, u_{n-1}', u_n^{(i)}) \neq 0 \qquad (i = 1, 2).$$

For, the condition that F and G_1 have a common root is given by the vanishing of the resultant R of (2) and (2′), the latter equation being written for $x_1' = 0$:

$$R(u_1, \cdots, u_{n-1}, x_1) = 0.$$

This function is analytic in the point $(0, \cdots, 0, \xi_1)$ and does not vanish identically, since g_1 and G_1 have no common factor in the origin.

Next, it is possible to join the points $(u_1', \cdots, u_{n-1}', u_n^{(1)})$ and $(u_1', \cdots, u_{n-1}', u_n^{(2)})$ by a curve lying in the neighborhood of the origin, in no point of which does G_1 vanish nor does $g_1/G_1 = A/B$. Hence a path on the configuration (2) is determined, along which $u_n^{(1)}$ is carried over into $u_n^{(2)}$.

(*b*) The function F cannot have a multiple factor,

$$P = u_n^q + B_1 u_n^{q-1} + \cdots + B_q.$$

For then the function $g_1 - x_1 G_1$ would also admit this factor in a point $(u_1, \cdots, u_n, x_1) = (0, \cdots, 0, \xi_1)$, where ξ_1 is in R_1^1:

$$g_1 - x_1 G_1 = P^l Q.$$

Hence

$$\frac{\partial}{\partial x_1}(g_1 - x_1 G_1) = P^{l-1} Q_1 = -G_1.$$

Thus $g_1 - x_1 G_1$ and G_1 would have a common factor in $(0, \cdots, 0, \xi_1)$, and consequently g_1 and G_1 would have a common factor in the origin. But this is contrary to hypothesis.

2. The Functions Ω_i

We turn now to the equations (*B*), for which $i = 2, \cdots, \mu$. Let

$$\omega_i(u_1, \cdots, u_n, x_i) = g_i(u_1, \cdots, u_n) - x_i G_i(u_1, \cdots, u_n),$$

and consider these functions on the manifold (2). It follows from the hypotheses that no G_i vanishes identically. Form the functions

$$\Omega_i(u_1, \cdots, u_{n-1}, x_1, x_i) = \prod_{k=1}^{p} \omega_i(u_1, \cdots, u_{n-1}, u_n^{(k)}, x_i) \quad (i = 2, \cdots, \mu),$$

where $u_n^{(k)}$ are the roots of (2). Then Ω_i is an algebroid polynomial in x_i, and furthermore it is of degree p, as we will now show.

Equation (2) has been shown to be irreducible. Its left-hand member was denoted by F. Let $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$, where $(u_1^0, \cdots, u_{n-1}^0)$ lies in the neighborhood of the origin and x_1^0 lies in R_1^1, be a point in which all the roots of F are distinct from one another. Then

$$F = \prod_{j=1}^{p} (u_n - f_j),$$

where $f_j(u_1, \cdots, u_{n-1}, x_1)$ is analytic in the point $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$ and actually contains x_1.

The coefficient of x_i^p in the expression for Ω_i is

$$\chi(u_1, \cdots, u_{n-1}, x_1) = (-1)^p \prod_{k=1}^{p} G_i(u_1, \cdots, u_{n-1}, u_n^{(k)}),$$

and we wish to show that this function does not vanish identically. If it did, then one of its factors, considered in the neighborhood of the point $(u_1^0, \cdots, u_{n-1}^0, x_1^0)$, must vanish identically. Let $k = 1$ and $j = 1$ correspond to this factor:

$$u_n^{(1)} = f_1(u_1, \cdots, u_{n-1}, x_1),$$

and let $u_n^0 = f_1(u_1^0, \cdots, u_{n-1}^0, x_1^0)$. Then $G_i(u_1, \cdots, u_n)$ must vanish whenever the function

$$u_n - f_1(u_1, \cdots, u_{n-1}, x_1),$$

which is analytic in the point $(u_1^0, \cdots, u_{n-1}^0, u_n^0, x_1^0)$ and vanishes there, and moreover, is irreducible there, vanishes. Hence $G_i(u_1, \cdots, u_n)$ must be divisible by this function at the point in question. But this is impossible, since f_1 actually contains x_1, while G_i does not.

The functions Ω_i are readily seen to be analytic at every point of the manifold

$$u_k = 0, \quad k = 1, \cdots, n-1; \quad 0 < |A - Bx_1| \leqq \infty, \quad |x_j| \leqq \infty \quad (j = 2, \cdots, n),$$

except at those points in which $x_i = \infty$, and in the neighborhood of such a point the function

$$\Omega_i'(u_1, \cdots, u_{n-1}, x_1, x_i') = x_i'^{-p}\, \Omega_i(u_1, \cdots, u_{n-1}, x_1, x_i) \quad \left(x_i' = \frac{1}{x_i}\right),$$

is seen to be analytic. For, in a point $(u_1, \cdots, u_{n-1}, x_1)$ in which each irreducible factor of (2) has all its roots distinct, Ω_i is evidently analytic. The excepted points form a manifold of a lower order of dimensions, and in its points Ω_i is continuous. Hence, by the extension of Riemann's theorem for removable singularities of functions of a single variable,* Ω_i is analytic in these points also.

3. The Function $\Phi(x_1, x_2)$

It may happen that the function $\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2)$ is divisible in the point $(0, \cdots, 0, \xi_1, \xi_2)$, where (ξ_1, ξ_2) is a point of the region R_1^2, by a function $f(u_1, \cdots, u_{n-1})$. Let the other factor, which shall not be divisible by such a function, be developed into a series of homogeneous polynomials in $(u_1, \cdots, u_{n-1})$, the coefficients being functions of (x_1, x_2) analytic in the region R_1^2; and let r be the order of the first of these terms. Then, on making if necessary a linear transformation of $(u_1, \cdots, u_{n-1})$, we can ensure the presence of the term in u_{n-1}^r. Let its coefficient be $\Phi(x_1, x_2)$:

$$\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2) = f(u_1, \cdots, u_{n-1})\{\Phi(x_1, x_2)u_{n-1}^r + (u_1, \cdots, u_{n-1})\},$$

where the parenthesis, when developed into a power series in $u_1, \cdots, u_{n-1}$, contains no terms of lower than the rth order, and the term in u_{n-1}^r is lacking.

The function $\Phi(x_1, x_2)$ is a polynomial in x_2 which does not vanish identically, and the coefficients are each analytic in R_1^1, the degree of Φ in x_2 being $\lambda \leqq p$. Hence $\Phi(x_1, x_2)$ itself is analytic at all points of R_1^2 for which $|x_2| < \infty$, and $x_2^{-\lambda}\Phi(x_1, x_2)$ is analytic at the remaining points of R_1^2.

A necessary condition for a simultaneous solution of the first two equations (A) is the following:

$$\Phi(x_1, x_2)u_{n-1}^r + (u_1, \cdots, u_{n-1}) = 0. \tag{4}$$

For, if $(u_1', \cdots, u_n', x_1', x_2')$ be such a solution, then $\Omega_2(u_1', \cdots, u_{n-1}', x_1', x_2') = 0$. Suppose $f(u_1', \cdots, u_{n-1}') = 0$. It is possible to find a point $(u'') = (u_1'', \cdots, u_n'')$ indefinitely near to (u') for which $f(u_1'', \cdots, u_{n-1}'') \neq 0$. The corresponding values $x_1 = x_1''$, $x_2 = x_2''$ lie near to x_1', x_2', and for $(u_1'', \cdots, u_{n-1}'', x_1'', x_2'')$ equation (4) is satisfied. From the continuity of the left hand side of (4) it follows, then, that (4) is also satisfied in $(u_1', \cdots, u_{n-1}', x_1', x_2')$.

We distinguish two cases: Case 1, $r = 0$; Case 2, $r > 0$.

Conversely, every solution of (4) yields a solution of the first two equations (B), but not necessarily a solution of the corresponding equations (A). For, it makes $\Omega_2 = 0$, and hence

$$\omega_2(u_1, \cdots, u_{n-1}, u_n^{(k')}, x_1, x_2) = 0,$$

* Cf. *Madison Colloquium*, p. 163.

and (2) is also satisfied by $u_n = u_n^{(k')}$. Indefinitely near to a given solution of (4) lies, however, a second solution of (4) which does yield a solution of (A).

4. Case 1, $r = 0$

In this case we have

$$\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2) = f(u_1, \cdots, u_{n-1})\{\Phi(x_1, x_2) + (u_1, \cdots, u_{n-1})\},$$

and (4) becomes

$$\Phi(x_1, x_2) + (u_1, \cdots, u_{n-1}) = 0, \tag{5}$$

where the ( ) vanishes for $u_1 = 0, \cdots, u_{n-1} = 0$.

If $(x) = (\xi)$ is a point not lying on $\mathfrak{S}_1$ and such that

$$\Phi(\xi_1, \xi_2) \neq 0,$$

then it is clear that a neighborhood of (ξ) and a region $\mathfrak{T}$ can be so chosen that, when (x) lies in the first region and (u) in the second, equation (5) cannot be satisfied. Hence for no such pair of points (x), (u) can (A) be satisfied, and consequently (ξ) is not a point of $\mathfrak{M}$. It follows, then, that the manifold $\mathfrak{M}$ is contained in the hyperplane $\mathfrak{S}_1$ and the points of the cylinder

$$x_1 = \xi_1, \quad x_2 = \xi_2, \quad |x_j| \leqq \infty \quad (j = 3, \cdots, n), \tag{6}$$

where

$$\Phi(\xi_1, \xi_2) = 0.$$

In case $\xi_2 = \infty$, $\Phi(x_1, x_2)$ is to be replaced by

$$\Phi'(x_1, x_2') = x_2^{-\lambda}\Phi(x_1, x_2) \qquad \left(x_2' = \frac{1}{x_2}\right).$$

We will now show that certain elements (6) of the cylinder

$$\Phi(x_1, x_2) = 0 \tag{7}$$

contain at least one point of $\mathfrak{M}$. To do this we will show that the degree λ of Φ in x_2 is positive.

Let ξ_1 be chosen arbitrarily. Ω_2 is a polynomial in x_2 of degree p, the coefficient of x_2 being $\chi(u_1, \cdots, u_{n-1}, x_1)$. We can, then, find in an arbitrary neighborhood of ξ_1 a point ξ_1', and, independently of the choice of ξ_1', in an arbitrary neighborhood of the origin a point $(u_1', \cdots, u_{n-1}')$ such that

$$\chi(u_1', \cdots, u_{n-1}', \xi_1') \neq 0,$$

and hence

$$f(u_1', \cdots, u_{n-1}') \neq 0.$$

Let $\phi(x_1)$ be the coefficient of x_2^λ in $\Phi(x_1, x_2)$. We restrict ξ_1' furthermore so that

$$\phi(\xi_1') \neq 0.$$

Let x_2' be a root of the equation

$$\Omega_2(u_1', \cdots, u_{n-1}', \xi_1', x_2) = 0,$$

and hence of the equation

$$\Phi(\xi_1', x_2) + (u_1', \cdots, u_{n-1}') = 0. \tag{8}$$

We now infer that $\lambda > 0$. For, if λ were $= 0$, we should have

$$\Phi(\xi_1', x_2) = \phi(\xi_1') \neq 0,$$

and by choosing $(u_1', \cdots, u_{n-1}')$ sufficiently near the origin, the second term in (8) could be made indefinitely small. This leads to a contradiction.

The roots of the equation

$$\Phi(\xi_1', x_2) + (u_1, \cdots, u_{n-1}) = 0 \tag{9}$$

when $(u_1, \cdots, u_{n-1})$ is taken at the origin, coincide with the λ roots of the equation

$$\Phi(\xi_1', x_2) = 0. \tag{10}$$

Hence, for all points $(u_1, \cdots, u_{n-1})$ of a certain neighborhood of the origin, the roots of the former equation lie near those of the latter.

Let the positive numbers η_i, $i = 1, \cdots, n$, be chosen at pleasure, and let ξ_1' be a point of the neighborhood of ξ_1 such that $\phi(\xi_1') \neq 0$. Then ξ_1' and a point $(u_1^0, \cdots, u_{n-1}^0)$ can be so determined that the corresponding roots u_n^0 of (2), x_1 being set $= \xi_1'$, will be distinct, and each will yield a point for which

$$|u_i^0| < \eta_i \qquad (i = 1, \cdots, n).$$

Moreover, if ξ_2' be an arbitrary root of equation (10) and an arbitrarily small neighborhood of ξ_2' be chosen, the point $(u_1^0, \cdots, u_{n-1}^0)$ can then be so taken that a root of (9) will lie in this neighborhood.

Furthermore, the point $(u_1^0, \cdots, u_{n-1}^0)$ can be so determined that, no matter what root of (2) be associated with it, the functions g_1, G_1, and likewise the functions g_2, G_2, will not both vanish in $(u^0) = (u_1^0, \cdots, u_n^0)$.

Lastly, the point $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$ can be so taken that, for all points $(u_1, \cdots, u_{n-1}, x_1)$ in its neighborhood, the roots x_2 of (9), as well as the roots u_n of (2), can be grouped together so as to constitute functions each analytic in $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$.

We are led, then, to the following result.

In the region of the space of the variables $(u_1, \cdots, u_{n-1}, x_1)$ *for which* $(u_1, \cdots, u_{n-1})$ *lies in the neighborhood of* $(u_1^0, \cdots, u_{n-1}^0)$ *and* x_1 *in the neighborhood of* ξ_1', *the* p *roots* u_n *of* (2) *are analytic functions.*

Each of these, when substituted in the second of the equations (*B*) *or* (*A*) *yields a function* x_2 *likewise analytic in the point* $(u_1^0, \cdots, u_{n-1}^0, \xi_1')$.

The systems of values $(u_1, \cdots, u_n, x_1, x_2)$ *thus found satisfy the first two of*

the equations (A) and (B), and they exhaust all such systems for which $(u_1, \cdots, u_{n-1})$ *lies in the neighborhood of* $(u_1^0, \cdots, u_{n-1}^0)$ *and* x_1 *in the neighborhood of* ξ_1'.

Equation (5) is an algebroid equation in x_2. Its left-hand side cannot have two distinct irreducible factors, each being of the form

$$E_0 x_2^l + E_1 x_2^{l-1} + \cdots + E_l, \qquad (0 < l),$$

where $E_k = E_k(u_1, \cdots, u_{n-1}, x_1)$ is analytic in each point $(0, \cdots, 0, \xi_1)$ for which ξ_1 lies in R_1^1. For, the roots, x_2, of this equation are precisely the values that the function

$$x_2 = \frac{g_2(u_1, \cdots, u_n)}{G_2(u_1, \cdots, u_n)}$$

takes on in the points of the configuration (2), and since the latter is irreducible, there must be a path in the $(u_1, \cdots, u_{n-1}, x_1)$-space along which a given root of (5) is carried over into any second root of (5).

The locus (7) consists conceivably in part of right lines,—or hyperplanes, if interpreted in the (x)-space,—

$$x_1 = \text{const.},$$

and these may conceivably cluster about the line $x_1 = A/B$,—or the hyperplane $\mathfrak{S}_1$.

The remainder of the locus consists of a finite number of monogenic analytic configurations

C_1: $$x_2 = \psi(x_1),$$

where $\psi(x_1)$ is a finitely multiple-valued function of x_1 having at most ordinary branch-points and poles in R_1^1. If $\Phi_1(x_1, x_2)$ be the product of the distinct irreducible factors of $\Phi(x_1, x_2)$ regarded as an algebroid polynomial in x_2, each factor being taken as primitive, then the totality of the configurations C_1 is also represented by the equation

$$\Phi_1(x_1, x_2) = 0.$$

From the foregoing we infer that *if* (ξ_1, ξ_2) *be any point of* C_1, *then the element*

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_j| \leqq \infty \qquad (j = 3, \cdots, n),$$

contains at least one point of $\mathfrak{M}$.

For, this element is a closed manifold in the space of analysis, and in every neighborhood of this element there are points of M, no matter how far $\mathfrak{T}$ be restricted. We have shown, namely, that a point (u) can be found indefinitely near the origin, for which the first two functions $\phi_i(u_1, \cdots, u_n)$ of (A) are defined and yield values of x_1, x_2 indefinitely near ξ_1, ξ_2 respectively. This point (u) can then be slightly modified, if necessary, so that all the functions ϕ_i will be analytic there.

The Algebraic Character of the Configurations C_1. Let x_1 and x_2 interchange their roles, *i. e.*, let the second equation (B) be solved for u_n, thus giving an equation

$$u_n^{p'} + A_1'(u_1, \cdots, u_{n-1}, x_2)\, u_n^{p'-1} + \cdots = 0. \tag{2'}$$

By the foregoing considerations we are led to a function

$$\Omega_2'(u_1, \cdots, u_{n-1}, x_2, x_1) = f'(u_1, \cdots, u_{n-1})\{\Phi'(x_2, x_1)\, u_n^{r'} + (u_1, \cdots, u_{n-1})\},$$

and we infer at once that $r' = 0$. For otherwise it would be possible to choose a point (x_1, x_2) not on $\mathfrak{S}_1$ or the locus (7), and also not on

$$\mathfrak{S}_1': \qquad A' - B' x_2 = 0$$

or on the locus

$$\Phi'(x_2, x_1) = 0, \tag{11}$$

and then determine a point (u) arbitrarily near the origin, for which the first two of the equations (A) admit a simultaneous solution. But here is a contradiction.

From the foregoing theory, the locus (11) may consist in part of right lines $x_2 = \text{const.}$ It will surely consist in part of a finite number of monogenic analytic configurations

$$C_1': \qquad x_1 = \psi_1(x_2),$$

where ψ_1 is finitely multiple-valued and has at most ordinary branch-points and poles in the region $R_1'^1$.

Combining these two results we see that a given configuration C_1 may reduce to a straight line $x_2 = \text{const.}$ If this is true of every configuration C_1, then each configuration C_1' must reduce to a straight line $x_1 = \text{const.}$

If, however, there is a configuration C_1 not such a right line, then C_1 and a certain C_1' must be the same monogenic analytic configuration; for, each element (6), (ξ_1, ξ_2) being a point of C_1, contains a point of $\mathfrak{M}$, and if $x_2 = \xi_2 \neq x_2'$ is not one of the lines which form part of the locus (11), then the above element must lead to a point (ξ_1, ξ_2) of some C_1'.

The configuration C_1 is algebraic. The curve is known to have at most an ordinary singularity* at any one of its points save, perhaps, the point (x_1', x_2'), in which the lines

$$A - Bx_1 = 0, \qquad A' - B' x_2 = 0$$

* A curve, surface, etc., is said to have an *ordinary singularity* at a point A if in the neighborhood of A it consists of one or more of the manifolds which are given by a set of equations of the form

$$G_k(x_1, \cdots, x_n) = 0 \qquad (k = 1, \cdots, \nu < n),$$

where G_k is analytic at A and vanishes there, but does not vanish identically.

intersect. For values of x_1 in the neighborhood of x_1', but distinct from this point, the equation of C_1 is

$$\Phi_0(x_1, x_2) = 0,$$

where Φ_0 is an algebroid polynomial in x_2 whose coefficients are analytic in the neighborhood of x_1' with the possible exception of this point. Moreover, on making if necessary a linear transformation of x_2, all the roots x_2 of this equation will remain finite. For, if $\xi_2 \neq x_2'$ be a finite cluster-point of roots x_2 when x_1 approaches x_1', then the point $(x_1, x_2) = (x_1', \xi_2)$ is a point of C_1', and C_1' is given near this point by the equation

$$\Phi_0'(x_2, x_1) = 0,$$

Φ_0' being analytic at $(x_2, x_1) = (\xi_2, x_1')$. Hence the coefficient of the highest power of x_2 in $\Phi_0(x_1, x_2)$ can be reduced to unity, and the other coefficients remain finite at x_1'. They have, therefore, at most removable singularities there, and thus the statement is proven.

The above investigation includes the case that the ( ) in (5) vanishes identically. Let (u^0) be a point of the neighborhood of the origin, in which neither G_1 nor G_2 vanishes. Then x_1 and x_2, computed for any point (u) in the neighborhood of (u^0), will necessarily make $\Phi(x_1, x_2)$ vanish. Consequently, if x_1 and x_2 are replaced in $\Phi(x_1, x_2)$ by g_1/G_1 and g_2/G_2 respectively, the resulting function of $(u_1, \cdots, u_n)$ vanishes identically in those variables. Hence g_1/G_1 and g_2/G_2 are connected by an algebraic relation.

More generally, if the left-hand side of (5) is divisible by an irreducible algebroid polynomial in x_2 with coefficients in x_1 alone, analytic in R_1^1, or by a power of such an algebroid polynomial, the other factor is of the form

$$\theta(x_1) + (u_1, \cdots, u_{n-1}),$$

where $\theta(x_1)$ is analytic in R_1^1 and the ( ) vanishes at the origin. For, this factor is an algebroid polynomial in x_2 of degree 0.

The Manifold $\mathfrak{C}_1$. The points (x_1, x_2) which lie on the manifolds C_1 and C_1' shall constitute the manifold $\mathfrak{C}_1^2$, and $\mathfrak{C}_1$ *shall consist of the points*

$$\mathfrak{C}_1: \qquad x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_j| \leqq \infty \qquad (j = 3, \cdots, n),$$

where (ξ_1, ξ_2) traces out $\mathfrak{C}_1^2$.

Those points of the cylinder $\mathfrak{C}_1$, for which (ξ_1, ξ_2) is a fixed point form an *element* of $\mathfrak{C}_1$.

THEOREM. *In Case* 1, $r = 0$, $\mathfrak{M}$ *lies on* $\mathfrak{C}_1$, *and each element of* $\mathfrak{C}_1$ *contains at least one point of* $\mathfrak{M}$.

For, no point of $\mathfrak{M}$ can lie in R_2 or R_2'. The only remaining points are those which lie on $\mathfrak{C}_1$ and those for which (x_1, x_2) is at the intersection of one of the

exceptional lines $x_1 = \text{const.}$ (i. e., $\mathfrak{S}_1$ or $\phi(x_1) = 0$) with a similar line, $x_2 = \text{const.}$ But $\mathfrak{M}$ is a connected manifold, and hence this case is excluded.

5. Case 2, $r > 0$

The Manifold $\mathfrak{S}_2$ and the Regions R_2, $\bar{R}_2$, R_2^k, $\bar{R}_2^k$. The manifold $\mathfrak{S}_2$ shall be defined as consisting of those points of R_1 in which Φ vanishes,

$$\mathfrak{S}_2: \qquad \Phi(x_1, x_2) = 0.$$

The remaining points of R_1 constitute the region R_2.

Let $\mathfrak{S}_2$ be imbedded in a region Σ_2 and let the points of Σ_2 which lie in $\bar{R}_1$ be removed from this region. The remainder of $\bar{R}_1$ forms the region $\bar{R}_2$, which shall be taken as *closed.*

The definitions of R_2^k, $\bar{R}_2^k$ are now given precisely as in the earlier case for R_1, $\bar{R}_1$.

The Equation

$$\Phi(x_1, x_2)\, u_{n-1}^r + (u_1, \cdots, u_{n-1}) = 0. \tag{12}$$

Let (ξ_1, ξ_2) be a point of R_2^2. Then equation (12) is equivalent to the following:

$$u_{n-1}^r + B_1 u_{n-1}^{r-1} + \cdots + B_r = 0, \tag{13}$$

$$B_k = B_k(u_1, \cdots, u_{n-2}, x_1, x_2), \qquad B_k(0, \cdots, 0, x_1, x_2) = 0,$$

where (x_1, x_2) lies, to begin with, in the neighborhood of (ξ_1, ξ_2).

It is shown as in the case of equation (2) that $B_k(u_1, \cdots, u_{n-2}, x_1, x_2)$ is analytic at every point of the manifold

$$u_k = 0, \qquad k = 1, \cdots, n-2; \qquad (x_1, x_2) \text{ in } R_2^2.$$

Let (ξ_1, ξ_2) be an arbitrary point of R_2^2. It is then possible to find a point (u^0) within an arbitrarily preassigned neighborhood of the origin, for which the first two of the equations (B), written for $x_1 = \xi_1$, $x_2 = \xi_2$, are satisfied. Thus (u^1) can be chosen near (u^0) so that no pair of functions g_i, G_i vanish in (u^1), the corresponding point (ξ_1', ξ_2') lying near (ξ_1, ξ_2). Hence for (u^1) equations (A) all have a meaning, and *thus it appears that there is at least one point of $\mathfrak{M}$ on the manifold*

$$x_1 = \xi_1, \qquad x_2 = \xi_2.$$

Since $\mathfrak{M}$ is perfect, the restriction that (ξ_1, ξ_2) be a point of R_2^2 can now be removed, and *the result just obtained is seen to hold when the point (ξ_1, ξ_2) is wholly arbitrary.*

6. The Case $\mu = 2$

When $\mu = 2$, the last $n - 2$ equations (A) each give for x_i a definite limiting value a_i, no matter how (u) approaches the origin, and we are thus led to the following result:

THEOREM. *When $\mu = 2$ and $r = 0$, the manifold consists of the intersection of the (reducible or irreducible) algebraic cylinder $\mathfrak{C}_1$ with the hyperplanes $x_j = a_j$, $j = 3, \cdots, n$.*

In particular, if $n = 2$, r is necessarily 0, and $\mathfrak{M}$ is the algebraic plane curve $\mathfrak{C}_1^2 = \mathfrak{C}_1$.

When $\mu = 2$ and $r > 0$, $\mathfrak{M}$ consists of the linear manifold

$$x_j = a_j \qquad (j = 3, \cdots, n).$$

7. THE GENERAL METHOD

We have discussed all cases in which $\mu = 1$ and $\mu = 2$, and the treatment of the general case, $\mu = \mu$, is already clearly indicated. When $\mu > 2$, however, there are still points requiring further development.

The argument here is as follows: Let (ξ_1, ξ_2) be a point of R_2^2, and let $r > 0$. Then (12) can be solved for u_{n-1} by (13). Moreover, a point (ξ_1', ξ_2') can be found near (ξ_1, ξ_2) and a point $(u_1', \cdots, u_{n-2}')$ near the origin such that one branch of (13),

$$u_{n-1} = f(u_1, \cdots, u_{n-2}, x_1, x_2)$$

will be analytic at $(u_1', \cdots, u_{n-2}', \xi_1', \xi_2')$ and

$$\frac{\partial f}{\partial x_2} \neq 0$$

there. Hence the last equation can be solved for x_2:

$$x_2 = \lambda(u_1, \cdots, u_{n-1}, x_1),$$

where λ is analytic at $(u_1', \cdots, u_{n-1}', \xi_1')$.

It is now possible to find a point $(u_1'', \cdots, u_{n-1}'', \xi_1'')$ near $(u_1', \cdots, u_{n-1}', \xi_1')$ such that, no matter what root u_n'' of (2) be associated with it, (A) will be defined in (u''). On the other hand, (ξ_1'', ξ_2'') lies near (ξ_1, ξ_2).

We begin as in the case $\mu = 2$ by forming the functions Ω_i and $\Phi(x_1, x_2)$, and we distinguish the two cases, $r = 0$ and $r > 0$. For $r = 0$, the discussion of § 4 is complete.

It may happen that for every pair of the μ equations in question, $r = 0$. Then $\mathfrak{M}$ lies at once on each of a set of (reducible or irreducible) algebraic cylinders

$$\Phi_k(x_k, x_1) = 0, \qquad (k = 2, \cdots, \mu),$$

and has at least one point in each element of such a cylinder. Moreover,

$$x_j = a_j, \qquad (j = \mu + 1, \cdots, n).$$

Hence $\mathfrak{M}$ lies on a finite number of irreducible algebraic curves in the (x)-

space. That $\mathfrak{M}$ consists of all the points of a certain number of these can be shown by a linear transformation. The details are indicated in the treatment of the special case of § 8.

When, on the other hand, for some pair of the μ equations $r > 0$, let these be the first two equations. Form the functions

$$\mathrm{X}_j(u_1, \cdots, u_{n-2}, x_1, x_2, x_j) = \prod_{k=1}^{r} \Omega_j(u_1, \cdots, u_{n-2}, u_{n-1}^{(k)}, x_1, x_j) \qquad (j = 3, \cdots, n),$$

where $u_{n-1}^{(k)}$ denotes a root of (12) or (13). On making, if necessary, a linear transformation of $u_1, \cdots, u_{n-2}$ we can write

$$\mathrm{X}_3(u_1, \cdots, u_{n-2}, x_1, x_2, x_3) = f_1(u_1, \cdots, u_{n-2})\{\Psi(x_1, x_2, x_3)u_{n-2}^s + (u_1, \cdots, u_{n-2})\},$$

where the $\{\ \}$ admits no factor in $u_1, \cdots, u_{n-2}$ alone, and the $(\ )$, when developed into a power series, contains only terms of at least the sth dimension, the term in u_{n-2}^s not appearing.

This equation is the precise analogue of the earlier equation $\Omega_2(u_1, \cdots, u_{n-1}, x_1, x_2) = 0$, and the treatment follows exactly the same lines.

First, X_3 is seen to be a polynomial in x_3 of degree pr, the coefficients being analytic in the points $(u_1, \cdots, u_{n-2}, x_1, x_2) = (0, \cdots, 0, \xi_1, \xi_2)$, where (ξ_1, ξ_2) is any point of R_2^2. $\Psi(x_1, x_2, x_3)$ is also a polynomial in x_3 with coefficients which are analytic in R_2^2.

A necessary condition that the first three equations (A) admit a simultaneous solution $(u_1, \cdots, u_n, x_1, x_2, x_3)$ is that

$$\Psi(x_1, x_2, x_3)u_{n-2}^s + (u_1, \cdots, u_{n-2}) = 0.$$

Again, we distinguish two cases: Case 1, $s = 0$; Case 2, $s > 0$.

In Case 1 it is shown as before that Ψ is of positive degree in x_3. The algebroid polynomial Ψ may admit factors which depend on x_1 and x_2 only. If these are suppressed and the remaining factor is denoted by Ψ_1, then the equation

$$\Psi_1(x_1, x_2, x_3) = 0 \tag{14}$$

defines a finite number of monogenic analytic configurations

$$C_2: \qquad x_3 = \psi(x_1, x_2),$$

where ψ denotes a finitely multiple-valued function having at most ordinary singular points* when (x_1, x_2) lies in R_2^2, as is shown by the equation (14).

Denote the coefficient of the highest power of x_3 in $\Psi(x_1, x_2, x_3)$ by $\phi(x_1,$

* A *function* is said to have an *ordinary singular point* if the corresponding analytic configuration has an ordinary singular point.

x_2). Then ϕ is analytic at every point of R_2^2. Denote the locus $\phi = 0$ in the space of the variables $(x_1, \cdots, x_n)$ by

$$S_2: \qquad \phi(x_1, x_2) = 0,$$

and in the $(x_1, \cdots, x_k)$-space by S_2^k. S_2 is regular at every point of R_2.

Let (ξ_1, ξ_2, ξ_3) be a point of C_2. Then the element

$$x_i = \xi_i, \quad i = 1, 2, 3; \qquad |x_j| \leqq \infty \qquad (j = 4, \cdots, n),$$

contains at least one point of $\mathfrak{M}$. Furthermore, if (ξ_1, ξ_2) lies in R_2^2, but not on S_2^2, the only points of $\mathfrak{M}$ are those which belong to C_2.

If $\psi(x_1, x_2) = \text{const.}$, then all points of the corresponding hyperplane, $x_3 = \text{const.}$, which is surely algebraic, belong to $\mathfrak{M}$, when $n = 3$, and lead to points of $\mathfrak{M}$ when $n > 3$.

Suppose ψ actually involves at least one of the letters x_1 and x_2, say x_2. We will now allow x_2 and x_3 to permute their rôles. We get, then, a function X_3' with an s', and it is seen as in the earlier case that s' must vanish. Thus we are led to a configuration

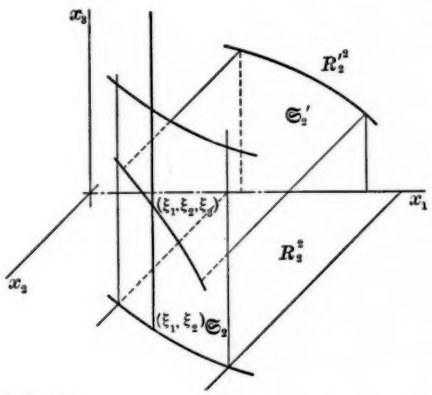
$$C_2': \qquad x_2 = \psi'(x_1, x_3),$$

and this must be the same as C_2.

It remains to show that C_2 is algebraic. We know already that C_2 has at most an ordinary singularity at any one of its points which does not lie on $\mathfrak{S}_1$ or $\mathfrak{S}_2$ and $\mathfrak{S}_1'$ or $\mathfrak{S}_2'$. Assume to begin with that $n = 3$. Let (ξ_1, ξ_2, ξ_3) be a (finite or infinite) point of intersection of C_2 with $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ (hence not lying in $\mathfrak{S}_1$ or $\mathfrak{S}_1'$), and let $\xi_1 \neq \alpha$, where

$$x_1 = \alpha \tag{15}$$

is the equation of one of the planes, if such exist, which form part of the surface $\mathfrak{S}_2$ or $\mathfrak{S}_2'$. The line



$$x_1 = \xi_1, \qquad x_2 = \xi_2 \tag{16}$$

meets the surface $\mathfrak{S}_2'$ in p' points. Surround each of these points by a neighborhood, $\sigma_1, \sigma_2, \cdots$. The line may meet C_2' outside of these neighborhoods. If so, each such point of intersection will be at most an ordinary singular point of C_2'. In general, the number of such points of intersection will be finite; in particular, the whole line (16) may lie in C_2'.

We may assume in the first case that all the points of intersection of the line in question with the surfaces $\mathfrak{S}_2'$ and C_2' lie in the finite region, for a suitable transformation of the group of the space of analysis will ensure this result.

Consider, then, the x_3 coördinate of C_2, regarded as a function of x_1 and x_2 in the neighborhood of the point $(x_1, x_2) = (\xi_1, \xi_2)$. This function is p-valued, and it has at most ordinary singularities, except possibly for the points (x_1, x_2) which lie on $\mathfrak{S}_2$. Moreover, all its determinations remain finite. Hence it satisfies an equation of the form

$$x_3^p + C_1(x_1, x_2)x_3^{p-1} + \cdots + C_p(x_1, x_2) = 0, \tag{17}$$

where $C_k(x_1, x_2)$ is analytic at all points of the neighborhood of (ξ_1, ξ_2) except possibly at such as lie on $\mathfrak{S}_2$ and where, furthermore, $C_k(x_1, x_2)$ remains finite in this region. It follows, then, from the extension of Riemann's theorem relating to removable singularities that $C_k(x_1, x_2)$ will be analytic throughout the entire neighborhood of (ξ_1, ξ_2) if properly defined there. Consequently C_2 or C_2' has only an ordinary singularity at (ξ_1, ξ_2, ξ_3).

If, on the other hand, the whole line (16) lies in C_2', we infer at once that it must be an isolated line of this nature. For, at any one of its points (ξ_1, ξ_2, ξ_3') not on $\mathfrak{S}_2'$, C_2' is given by an equation of the form

$$G(x_1, x_2, x_3) = 0,$$

where G is analytic at this point and vanishes there and is irreducible there, and moreover

$$G(\xi_1, \xi_2, x_3) \equiv 0.$$

Hence if G be developed into a power series in $x_3 - \xi_3'$, each coefficient will vanish at the point $(x_1, x_2) = (\xi_1, \xi_2)$. But the coefficients will not admit a common factor, and hence for no other point (ξ_1', ξ_2') of the neighborhood of (ξ_1, ξ_2) will $G(\xi_1', \xi_2', x_3)$ vanish identically.

In the neighborhood, then, of $(x_1, x_2) = (\xi_1, \xi_2)$ the function x_3 is given by an equation of the form (17), whose coefficients $C_k(x_1, x_2)$ are meromorphic in the neighborhood of (ξ_1, ξ_2) with the possible exception of this one point. According to a theorem proved by Hartogs* such a function $C_k(x_1, x_2)$ must

*Mathematische Annalen, vol. 70 (1911), p. 217. Cf. also *Madison Colloquium*, p. 165.

be meromorphic at (ξ_1, ξ_2), also, and hence the foregoing result is extended to all points of intersection of C_2 with $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ with the sole exception of such as lie in one of the planes (15).

If $\psi(x_1, x_2)$ does not depend on x_1, C_2 is a cylinder whose elements are parallel to the x_1-axis, and hence C_2 has at most ordinary singularities in the points where it meets the planes (15) and $\mathfrak{S}_1, \mathfrak{S}_1'$. Thus all points of the space of analysis are accounted for, and C_2 is seen to be algebraic.

If, however, $\psi(x_1, x_2)$ actually involves x_1, then the roles of x_1 and x_2 (or x_3) can be interchanged, and reasoning similar to the foregoing will show that $\psi(x_1, x_2)$ first has only ordinary singularities in R_1^2 except for isolated points. Secondly, these points are eliminated by Hartogs's theorem.

Finally, the points of intersection of the plane $\mathfrak{S}_1$ with the corresponding planes of the permuted variables are disposed of in a similar manner, and thus we see that the configuration C_2, in the extended space of analysis, has at most ordinary singular points. It is, therefore, algebraic.

The intersection of the part of $\mathfrak{S}_2$ distinct from the planes (15) with the analogous part of $\mathfrak{S}_2'$ may comprise curves not lying on C_2, and we are not able to say from the foregoing analysis whether these belong to $\mathfrak{M}$ or not. We shall return to this question in § 9.

8. The Final Theorem for the Case $\mu = 3$, $n = 3$

Beginning with the first two of the equations (B), we are led first to distinguish between the cases $r = 0$ and $r > 0$.

In Case 1, $r = 0$, the manifold $\mathfrak{M}$ lies wholly on an algebraic cylinder, which may be reducible.

On permuting the variables x_1, x_2, x_3 and the corresponding equations (B), and applying the results which have been obtained, it is seen that one must again distinguish two cases, according as the new r' is $= 0$ or > 0.

In the first case it appears that $\mathfrak{M}$ lies on a second algebraic cylinder, whose elements are perpendicular to those of the first, and hence $\mathfrak{M}$ lies on a finite number of irreducible algebraic space curves.

That the points of $\mathfrak{M}$ coincide with the totality of those of a certain number of these curves is seen from the fact that, on making if necessary a linear transformation

$$x_i' = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 \qquad (i = 1, 2, 3),$$

and introducing new functions g_i', G_i' such that

$$\frac{g_i'}{G_i'} = a_{i1} \frac{g_1}{G_1} + a_{i2} \frac{g_2}{G_2} + a_{i3} \frac{g_3}{G_3},$$

a point of $\mathfrak{M}$ will go over into a point of $\mathfrak{M}'$, and furthermore an element of

one of the cylinders containing $\mathfrak{M}'$ in the transformed space will meet $\mathfrak{M}'$ in general in only one point. But at least one point of $\mathfrak{M}$ must lie on each element of such a cylinder.

There will remain at most a finite number of elements of the cylinder in question which meet the curves on which $\mathfrak{M}$ lies in more than one point, but only in a finite number of points. These points also belong to $\mathfrak{M}$, since this manifold from its nature is perfect.

If, on the other hand, at least one of the numbers r, r' is positive, let it be r. Since we have assumed $n = 3$, s will always be $= 0$. The points of $\mathfrak{M}$ then coincide with the totality of those of a finite number of irreducible surfaces. For, if (ξ_1, ξ_2) is a point of R_2 not lying on S_2, the line

$$x_1 = \xi_1, \qquad x_2 = \xi_2, \qquad |x_3| \leqq \infty$$

meets the surfaces C_2 in a finite number of points, each of which is a point of $\mathfrak{M}$, and these are the only points of $\mathfrak{M}$ on such a line.

If, however, the above line lies in $\mathfrak{S}_1$ or $\mathfrak{S}_2$ or S_2, let two of the x's, as x_2 and x_3, be interchanged. A point (ξ_1, ξ_2, ξ_3) of the above line for which (ξ_1, ξ_3) lies in $R_2'^2$, but not on S_2', will belong to $\mathfrak{M}$ if, and only if, it lies on the surfaces in question.

Thus doubt remains only for such points as lie at once on $\mathfrak{S}_1$ or $\mathfrak{S}_2$ or S_2 and $\mathfrak{S}_1'$ or $\mathfrak{S}_2'$ or S_2'; i. e., on certain manifolds of a lower number of dimensions. The proof that no such points belong to $\mathfrak{M}$ unless they lie on the surfaces in question will be given in § 9.

Had we taken $\mu = 3$, $n = 4$, it might have happened that neither r nor s is 0. In that case, the points of $\mathfrak{M}$ have their first three coördinates wholly arbitrary, and the fourth is a constant:

$$x_4 = a_4 = \frac{g_4(0, 0, 0, 0)}{G_4(0, 0, 0, 0)}.$$

Thus $\mathfrak{M}$ is the hyperplane $x_4 = a_4$.

9. Proof of the Non-Existence of Manifolds of Lower Order of Dimensions*

Restricting ourselves still to the case of § 8, namely, $\mu = 3$, $n = 3$, we begin by disposing of any manifolds S_2 which may be present. For this purpose it is enough to apply a suitable linear transformation to x_1, x_2, x_3, replacing the functions ϕ_1, ϕ_2, ϕ_3 by new functions through the same transformation. Thus a cylinder S_2 goes over either into one of the new surfaces C_2 or else its points are seen not to belong to $\mathfrak{M}$, with the exception of such as lie on the curves which it is the object of this paragraph to investigate.

* This paragraph was written in June, 1918.

We have, then, a finite number of algebraic surfaces, whose totality is denoted by C_2 and all of whose points are points of $\mathfrak{M}$. No other points of the (x)-space can belong to $\mathfrak{M}$ except possibly those in which an $\mathfrak{S}_1$ or an $\mathfrak{S}_2$ cuts an $\mathfrak{S}_1'$ or an $\mathfrak{S}_2'$, and we proceed to show that these points are never points of $\mathfrak{M}$.

Suppose, then, that $\mathfrak{M}$ contained a point (x^0) lying on the intersection of $\mathfrak{S}_2$ with $\mathfrak{S}_2'$, but not on C_2, and suppose (x^0) to be an ordinary point of such a curve. Then we can arrange things so, by means of a linear transformation of the sort just considered, that (x^0) lies in the finite region and the plane

$$x_3 = x_3^0 \tag{18}$$

meets this arc in only one point near (x^0), and the same will be true of the intersection of every other plane

$$x_3 = \lambda, \tag{19}$$

where

$$|\lambda - x_3^0| < h,$$

h being a suitably chosen positive number.

Let us cut the (x)-space by the plane (19), considering now those points (x) given by (A) which lie in this plane. Analytically this means that we restrict (u) to lying on the surface

$$g_3(u_1, u_2, u_3) - \lambda G_3(u_1, u_2, u_3) = 0, \tag{20}$$

but not at a point of any one of the three curves

$$g_i(u_1, u_2, u_3) = 0, \qquad G_i(u_1, u_2, u_3) = 0 \qquad (i = 1, 2, 3). \tag{21}$$

For a non-specialized choice of the coördinate system in the (u)-space equation (20) is equivalent to an algebroid equation,

$$u_3^\nu + \Gamma_1 u_3^{\nu-1} + \cdots + \Gamma_\nu = 0, \tag{22}$$

$$\Gamma_k = \Gamma_k(u_1, u_2, \lambda), \qquad \Gamma_k(0, 0, \lambda) = 0.$$

We are thus led to a transformation

$$x_1 = \phi_1(u_1, u_2, u_3), \qquad x_2 = \phi_2(u_1, u_2, u_3), \tag{A'}$$

where x_1, x_2 are single-valued functions on the algebroid configuration (20), in general analytic; and we are interested in certain points of the manifold $\mathfrak{M}'$ belonging to (A').

It is impossible for x_1 and x_2 both to remain finite in the neighborhood of a point $\lambda = \lambda_0$:

$$|\lambda - \lambda_0| < \delta,$$

where λ_0 has a value near x_3^0:

$$|\lambda_0 - x_3^0| < h' < h.$$

For then x_1 and x_2 would both approach limits

$$x_1 = \psi_1(\lambda), \qquad x_2 = \psi_2(\lambda),$$

and hence $\mathfrak{M}$, for all values of x_3 near x_3^0, would be restricted to the one-dimensional locus

$$x_1 = \psi_1(x_3), \qquad x_2 = \psi_2(x_3).$$

But $\mathfrak{M}$ contains at least the points of intersection of the plane (19) with C_2, and this intersection is a curve.

Let $\mathfrak{M}$ be encased in a neighborhood Σ so chosen that the cross-section σ_1 of Σ by (19) near (x^0) is exterior to the remainder σ of the total cross-section. Let ϵ be so chosen that, when (u) lies in a region $\mathfrak{T}$, for whose points

$$|u_k| < \epsilon \qquad (i = 1, 2, 3). \tag{23}$$

(x) lies in Σ.

If, now, (x^0) be a point of $\mathfrak{M}$, it is possible to find a point (x') near (x^0) such that a point (u') of the above $\mathfrak{T}$ yields (x') through (A); and moreover $u_1' u_2'$ can here be restricted to being arbitrarily small:

$$|u_1'| < \epsilon_1, \qquad |u_2'| < \epsilon_1. \tag{24}$$

Furthermore, if λ be chosen arbitrarily within a suitable region

$$|\lambda - \lambda'| < \eta, \qquad \lambda' = x_3'; \tag{25}$$

and if u_1, u_2 be taken near u_1', u_2', then equation (20) will have a root u_3 near u_3', and (A) will be defined in the new point (u), the corresponding point (x) lying near (x').

Let S be a region of the (x_1, x_2)-plane including in its interior the region σ_1 which corresponds to a λ in (25), but not including any point of the σ which corresponds to such a λ. These conditions can be met by restricting Σ and h suitably at the outset. Then we can find a λ_1 in (25) and a point (u) for which (A) is defined, such that

$$(i) \qquad |u_1| < \epsilon_1, \qquad |u_2| < \epsilon_1;$$

$$(ii) \qquad (x_1, x_2) \text{ lies outside of } S.$$

And the same will be true if λ be chosen arbitrarily in a certain neighborhood of λ_1:

$$|\lambda - \lambda_1| < \eta_1, \tag{26}$$

which region lies wholly in (25).

Next, we can find a λ_2 in (26) and a point (u) for which (A) is defined, such that

(*i*) $$|u_1| < \epsilon_2, \qquad |u_2| < \epsilon_2;$$

(*ii*) (x_1, x_2) lies outside of S.

And the same will be true if λ be chosen arbitrarily in a certain neighborhood of λ_2:

(27) $$|\lambda - \lambda_2| < \eta_2,$$

which region lies wholly in (26).

Repeating the step indefinitely, we obtain a set of nested regions in the λ-plane, which have at least one point λ'' in common. Giving to λ this value, $\lambda = \lambda''$, we see that the plane

(28) $$x_3 = \lambda''$$

contains a set of points (x_1, x_2) all lying outside of S and corresponding to points (u) for which (A) is defined and $\lim(u) = (0)$. These points (x) have at least one point of condensation which lies outside of S, and this point belongs both to $\mathfrak{M}$ and to the $\mathfrak{M}'$ corresponding to $\lambda = \lambda''$.

We can, then, find a point (u'') on (20) arbitrarily near the origin such that (A) is defined in it and the point (x'') $(x_3'' = \lambda'')$ lies in Σ but outside the S corresponding to $\lambda = \lambda''$, and hence in the σ which corresponds to $\lambda = \lambda''$.

Returning to (x'), we now modify its choice so that $x_3' = \lambda''$; the point (x') still lying in σ_1 and (u') being restricted as before.

We now are able to deduce a contradiction. For, (u') and (u'') can be joined by a path lying on (20), at each point of which (A) is defined, and furthermore this path can be taken to lie wholly in (23). As (u) describes this path, (x) goes from (x') to (x''), always remaining in the plane (28), and hence (x) passes outside of σ_1 and σ; i. e., outside of Σ. But this is impossible, since each point (u) of (23) for which (A) is defined, yields a point (x) of Σ.

It follows, then, that $\mathfrak{M}$ can have no point which is an ordinary point of intersection of $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ not lying on (15). Other points of intersection of $\mathfrak{S}_2$ and $\mathfrak{S}_2'$ not lying on (15), being isolated, cannot belong to $\mathfrak{M}$, either.

To dispose of the points of (15), it is sufficient to allow x_1 to interchange its rôle with x_2 or x_3; and the points of $\mathfrak{S}_1$, $\mathfrak{S}_1'$ are disposed of in like manner.

The method admits extension to the higher cases. Thus, when $n = 4$ and $\mu = 4$, the manifold $C = C_3$ consists of a reducible or irreducible algebraic hypersurface (three-dimensional). Here, we cut by the pair of hyperplanes,

$$x_3 = \lambda_3, \qquad x_4 = \lambda_4.$$

If, on the other hand, $n = 4$ and the manifold of the maximum number of dimensions consists of a surface (two-dimensional), the case is analogous to that of the curve treated in § 8, or else to the linear manifold treated at the end of that paragraph.

10. The General Theorem

We can now state the result in the general case, the foregoing treatment applying without let or hindrance to the proof in that case.

THEOREM. *Let*

$$x_i = \frac{g_i(u_1, \cdots, u_n)}{G_i(u_1, \cdots, u_n)} \qquad (i = 1, \cdots, n),$$

be such a transformation as is defined in the introductory paragraph. The manifold $\mathfrak{M}$ *is then made up of a finite number of algebraic manifolds of the following kind.*

In the space of the first μ *variables,* $(x_1, \cdots, x_\mu)$, *where* $1 \leqq \mu \leqq n$, *there exists a manifold* $\mathfrak{K}$ *formed by a finite number of irreducible algebraic curves* $(k = 1)$, *surfaces* $(k = 2)$, *or hypersurfaces of order* $k < \mu$, *this number* k *being the same for all; or finally, when* $\mu < n$, $\mathfrak{K}$ *may include all the points of the space in question, and we set here* $k = \mu$.

Then $\mathfrak{M}$ *consists of the points* $(x_1, \cdots, x_n)$, *where* $(x_1, \cdots, x_\mu)$ *is an arbitrary point of* $\mathfrak{K}$, *and*

$$x_j = a_j = \frac{g_j(0, \cdots, 0)}{G_j(0, \cdots, 0)} \qquad (j = \mu + 1, \cdots, n).$$

We note that, if $\mathfrak{M}$ be imbedded in an arbitrarily restricted neighborhood $\mathfrak{U}$, then $\mathfrak{T}$ can be so chosen that the images (x) of all points (u) of $\mathfrak{T}$ will lie in $\mathfrak{U}$. For, if the points of $\mathfrak{U}$ be removed from the (x)-space, the remaining region will be closed. To each of its points (x') corresponds a definite positive η such that, if $\mathfrak{T}$ lie in the region

$$|u_k| < \eta \qquad (k = 1, \cdots, n),$$

(x') will not be the image of any point (u) of $\mathfrak{T}$. And now, if each η is chosen as large as possible, it is shown by familiar reasoning that the lower limit of the η's for the closed region in question is positive.

HARVARD UNIVERSITY,
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SPACE INVOLUTIONS DEFINED BY A WEB OF QUADRICS*

BY

VIRGIL SNYDER AND F. R. SHARPE

1. PROPERTIES OF THE INVOLUTION

1. **Statement of problem.** The following paper is concerned with the study of the transformation between the two spaces (x'), (x) which is defined by the equations

$$\rho x_i' = \phi_i(x) \qquad (i = 1, 2, 3, 4), \tag{1}$$

where $\phi_i(x) = 0$ is the equation of a quadric surface. It has been treated synthetically by Reye† but only incidentally in connection with line congruences. In the existing memoirs the involution of the whole of space is not considered except in the special case in which the system of quadrics have six common points; this special case has been extensively studied.‡

2. **Images of planes, lines, and points.** A plane s_1 in (x') goes into a quadric surface s_2 in (x), belonging to a definite web. A line c_1' in (x') has for image a space quartic c_4 of genus 1 in (x); the line is the basis of a pencil of planes, and the image quartic is the basis of the pencil of image quadric surfaces. A point P' in (x') has 8 image points in (x); the point P' is the vertex of a bundle of planes, and the image points are the eight basis points of the bundle of image quadric surfaces.

A plane s_1 in (x) goes into a Steiner surface s_4' in (x'), since by means of the equation of the plane, the coördinates of a point on the image surface are expressible as quadratic functions of three homogeneous parameters. Since the coefficients a_i in the equation $\sum a_i x_i = 0$ of a plane do not enter the equation of the image surface linearly, the system of Steiner surfaces in (x'), which appear as images of the whole system of planes of (x), is not linear.

* Presented to the Society, September 4, 1917.

† T. Reye, *Ueber die reciproke Verwandtschaft von F Systemen und ϕ^2-Geweben und die quadratischen F^2-Systeme achter Stufe*, Journal für die reine und angewandte Mathematik, vol. 82 (1876), pp. 173–206, and *Ueber Strahlensysteme zweiter Classe und die Kummersche Fläche vierter Ordnung mit sechzehn Knotenpunkten*, ibid., vol. 86 (1879), pp. 84–107.

‡ For the literature of this case, see Snyder, *An application of the (1, 2) quaternary correspondence to the Kummer and Weddle surfaces*, these TRANSACTIONS, vol. 12 (1911), pp. 354–366.

A line c_1 in (x) goes into a conic c_2' in (x'), since by means of the equations of c_1 the coördinates of a point on the image locus can be expressed as quadratic functions of two homogeneous parameters.

A point P in (x) has a single point P' in (x') for image; this appears directly from equation (1).

3. **Surfaces of coincidences and of branch points.** A point in (x') which has two coincident images in (x) is called a *branch-point*. Let P' be a branch-point and P the corresponding coincidence. To the bundle of planes through P' corresponds a bundle of quadrics having a common tangent line at P. Let $P \equiv (0, 0, 0, 1)$, and let $x_1 = 0$, $x_2 = 0$ be the equations of the common tangent line. We may take the equations of the three linearly independent quadrics through P in the forms $x_1 x_4 + f = 0$, $x_2 x_4 + \psi = 0$, $(a x_1 + b x_2) x_4 + \theta = 0$, wherein f, ψ, θ are quadratic forms in x_1, x_2, x_3. In this bundle is a cone having its vertex at P, hence we have the theorem:

THEOREM I: *The surface of coincidences is the locus of the vertices of the cones contained in the web of quadrics.*

The equation $\sum \lambda_i \phi_i = 0$ represents a cone when each of its first partial derivatives as to x_i vanishes. By eliminating λ_i from these equations we obtain the equation of the surface of coincidences; it is the jacobian of the web of quadrics. It will be denoted by K_4. The c_4 image of a straight line of (x') meets K_4 in 16 points, hence:

THEOREM II: *The locus of branch points is a surface of order* 16.

It will be denoted by L_{16}'. The complete image of L_{16}' is of order 32; it consists of K_4 counted twice, and of a residual surface of order 24, which we denote by R_{24}.

A quadric of the web meets K_4 in a c_8 of genus 9, hence a plane section of L_{16}' is of genus 9. A cone of the web meets K_4 in a c_8 of genus 8, having a double point at the vertex of the cone. The image plane meets L_{16}' in a curve of genus 8, having a double point at the image point of the vertex of the cone. It is a tangent plane to L_{16}', hence we may say:

THEOREM III: *The images of the quadric cones of the web are the tangent planes to* L_{16}'.

We shall now prove the theorem:

THEOREM IV: *The surface* K_4 *contains ten straight lines.* We have seen that in the web of quadrics are ∞^2 cones, the vertices of which lie on K_4. If a quadric of the web is composite, every point of its line of vertices lies on K_4; if (x) is any point on the line common to the two component planes, then for (x) every first minor of the discriminant of the quadric must vanish. In the system $\sum \lambda_i \phi_i = 0$ are ten composite quadrics.*

* Salmon, *Algebra*. Lesson 19 in the 4th edition, 1885.

4. **Particular lines.** Given a point P' in (x'). Its images in (x) are $P_1, P_2, \cdots, P_8$. The bundle of planes through P' goes into the bundle of quadrics through the eight associated points P_i. A straight line through P' is determined by one other point; a c_4 through all P_i is determined by one other point. If an additional point is chosen on the straight line $P_i P_k$, the associated c_4 has three collinear points, hence consists of the line $P_i P_k$ and a space cubic which meets the line in two points. The image of this composite quartic is a straight line in (x') which touches L'_{16} at the images of the points of intersection of $P_i P_k$ and its residual cubic. Thus, any line in (x) which joins two associated points has a line for image in (x'). Between these two lines exists a $(1, 2)$ correspondence, the double points of which are the points of intersection of $P_i P_k$ and K_4, not on the residual cubic. The lines $P_i P_k$ are all double tangents to R_{24}. Through any point P of (x) seven such bitangents can be drawn. The image lines are all bitangents to L'_{16}. Through any point P' of (x') twenty-eight such bitangents can be drawn, since there are 28 lines $P_i P_k$ associated with any point P'.

5. **Images of the ten lines on K_4.** It has been seen that the web of quadrics $\sum \lambda_i \phi_i = 0$ contains ten pairs of planes, and that the line of intersection of the planes of each pair lies on K_4. The image of each line γ_i is a conic γ'_i on L'_{16}. Since the pair of planes constitutes a quadric of the web, and the image of a quadric of the web is a plane, it follows that the plane of the conic is the image plane of the composite quadric. Since every point of the line is on K_4, the plane of the image conic touches L' at every point of the conic.

The curve of intersection of a quadric of the web with K_4 has for image a plane section of L'_{16}. In the case of a composite quadric, the curve of intersection with K_4 consists of the line of vertices taken twice and of two elliptic cubic curves. Hence the plane of each singular conic γ'_i meets L'_{16} in two elliptic curves, each of order six.

6. **Successive images of a plane.** A plane s_1 of (x) goes into a Steiner surface s'_4, having three concurrent double lines p'_1, p'_2, p'_3.

The complete image of s'_4 consists of s_1 and of a residual surface s'_7 of order 7. The plane s_1 meets K_4 in a plane quartic curve through which s'_7 passes; the residual intersection of s_1 and s'_7 consists of three lines p_1, p_2, p_3, images of the double lines of s'_4. The curve (s_1, K_4) goes over into a curve of order 8 which is a curve of contact of s'_4 and L'_{16}. The residual curve of intersection of s'_4 and L'_{16} is a c'_{48}; it is the image of the curve (s_1, R_{24}).

The congruence of lines $P_i P_k$ in (x) is of order 7 and class 3; it has R_{24} for focal surface.

A general line c_1 meets K_4 in 4 points; its image in (x') is a conic c'_2 which touches L'_{16} in the images of the points of intersection. The residual 24 points of intersection of c'_2 and L'_{16} are images of the points (c_1, R_{24}).

A line $P_i P_k$ meets K_4 in four points, two of which are on the residual c_3. The image line is bitangent to L'_{16} and meets it in 12 other points. The points of tangency are images of the points (c_1, c_3), and they have points of tangency of c_1 and R_{24} for residual images. Two of the residual points of c'_1, L'_{16} are coincidences on K_4; the remaining points have each a pair of intersections of c_1 with R_{24} for images. The same 10 points have for residual images the 10 residual intersections of c_3 with K_4.

7. **Two planes and their images.** Let p_1, p_2 be two planes in (x), and s'_1, s'_2 their image Steiner surfaces in (x'). The image of the line (p_1, p_2) is the conic p' common to both Steiner surfaces, which also intersect in a residual c'_{14} having 10 points on p'. To obtain further properties, pass a plane through a double line c'_1 of s'_1. It meets s'_1 in a conic c'_2 and s'_2 in a quartic c'_4. The line c'_1 meets s'_2 in four points $P'_1, \cdots, P'_4$; through P'_1 pass p' and c'_{14}; through each of the others c'_{14} passes twice. The curves c'_2, c'_4 meet in 8 points, through 7 of which c'_{14} passes, and the other lies on p'. The partial image of c'_1 is c_1 in p_1. The residual s'_7, image of s'_2, meets p_1 in a curve of order 7, having one point on (p_1, p_2). The other six are three pairs of images of P'_2, P'_3, P'_4. The plane through c'_1 goes into a quadric of the web through c_1. The plane p_1 cuts the quadric in c_1 and another line $\bar{c}_1$.

The plane p_2 meets the quadric in a conic c_2 which has a point on c_1 and one on $\bar{c}_1$. The image of $\bar{c}_1$ is the conic c'_2, and that of c_2 is c'_4. Since (p_1, p_2), $\bar{c}_1$, and c_2 have a point in common, it follows that p', c'_2, c'_4 have a point in common. The curve c'_{14} does not pass through this point. Since c'_{14} has three double points and one simple intersection on each double line of s'_1, s'_2, its genus is 6.

8. **Double curve and cuspidal curve on L'_{16}.** It has been seen that the locus of points in (x') which have two coincident images in (x) is L'_{16}, and that the locus of the coincidences is K_4. On L'_{16} is a cuspidal curve, each point of which has at least three coincident images and a double curve, each point of which has a pair of coincident images. The points of intersection of these two curves are either points to which correspond four coincident images, or a coincidence of three image points at one point and a simultaneous coincidence of two at another. The orders of these curves are the number of cusps and double points in a general plane section of L'_{16}.

Since the order and genus of a general plane section are known, all the characteristic numbers can be determined when one more is known. The bitangents of the plane section are the lines whose images in (x) are composite quartic curves. Of the bitangents, 10 are the lines in which the plane of section meets the ten singular planes. The image of the plane of section is a quadric of the web. If a line in (x') has for image in (x) a line and a residual cubic, the image line is a generator of the quadric and joins a pair of associated points P_i, P_k.

Such a line is a component of the basis curve of a pencil of quadrics contained in the web. Let $x_1 x_3 - x_2 x_4 = 0$ be the equation of a quadric. A generator of one system has equations $x_1 = \lambda x_2$; $\lambda x_3 = x_4$. Any quadric of the web has an equation of the form $\sum a_i \phi_i = 0$. Replace x_1 by λx_2, x_4 by λx_3, equate the coefficients of x_2^2, $x_2 x_3$, x_3^2 to zero and eliminate the a_i. The result is a sextic in λ, hence 6 generators of the λ-system join associated points. The same quadric contains 6 generators of the other stysem which join associated points.

Hence the total number of bitangents is 22. From Plücker's equations we now obtain $\delta = 60$, $\kappa = 36$, from which we may state the theorem:

THEOREM. *The surface L'_{16} has a double curve of order* 60 *and a cuspidal curve of order* 36.

9. **Contact and intersection curves on K_4.** When the point P' describes c'_{36} on L'_{16}, the triple image P is on K_4, and the remaining 5 images are on R_{24}. But P is also on R_{24}, and since a definite direction is associated with every point of K_4 (the limit of the direction $P_i P_k$ as $P_i \doteq P_k$) it follows that K_4 and R_{24} touch each other along the curve described by P. This curve is in (1, 1) correspondence with c'_{36}, and is of order 18. When P' is a point on c'_{60} on L'_{16}, two of its images coincide at P_1, two more at P_2, and there are four other images on R_{24}. P_1 is on K_4, hence P_2 is on R_{24}; but P_2 is on K_4, hence P_1 is on R_{24}. Hence K_4, R_{24} intersect along the curve described by P_1 and P_2 as P' describes c'_{60}. The locus of P_1 (and P_2) is of order 60; its image in (x') is c'_{60} counted twice.

10. **Special properties of γ_i, γ'_i.** The image of any straight line γ on K_4 is a singular conic γ' on L'_{16} (No. 5). The complete image of γ' consists of γ counted twice and of a residual cubic g_3 in each component plane π_1, π_2 of the composite quadric of the web belonging to γ. These two cubics both lie on R_{24}, but not on K_4. Each plane touches R_{24} along g_3. Each plane π_i meets K_4 in a cubic c_3, passing through the three points (γ, g_3), and meeting g_3 in six others. The image of each c_3 is a c'_6 in the plane π' of γ'. Together they constitute the residual intersection of π' with L'_{16}. Each c'_6 touches γ' in three points, images of the points (γ, c_3), and cuts it in six points, images of the intersections (g_3, c_3). The complete image of c'_6 is c_3 counted twice, a residual c_6 in π_1, and a c_{12} in π_2. Each plane π is tangent to the contact curve c_{18} in the three points (γ, c_3). The remaining 9 points of intersection of π with c_{18} (not on γ) are on c_3, hence c'_6 has 9 cusps. At each of the three contacts of c'_6 with γ' the plane π' meets the cuspidal curve c'_{36} in three points. At each intersection of c'_6 with γ' the plane π' touches the double curve c'_{60}; the residual intersections of c'_{60} with π' are at the 36 points common to the two sextics.

11. **Complete images of cuspidal and double curves.** The complete image

of c'_{36} is c_{18} counted three times and a residual c_{90} which is a cuspidal curve on R_{24}. The image of c'_{60} is c_{60} counted twice and a residual c_{120} which is a double curve on R_{24}. The curve c_{90} meets K_4 in 360 points. Let t of these points T be on c_{18} and c_{60}, that is, points at which four images of a point in (x') coincide; let there be m points M on c_{60} such that for each M the corresponding point in (x') has two images at M and three at an intersection of c_{18} and c_{60}. At the points T the curve c_{90} has contact with K_4, hence

$$2t + m = 360.$$

The curve c_{120} meets K_4 in 480 points. Let there be p points in (x') whose images in (x) consist of three coincidences at P_1, P_2, P_3, and of two residual points. We have the relation

$$3p + m = 480.$$

If we apply the Zeuthen formula for the number of coincidences in a multiple correspondence to the $(2, 1)$ correspondence between c_{60}, c'_{60} and the $(4, 1)$ between c_{120}, c'_{60} we obtain the relation

$$p = t.$$

Hence

$$p = t = m = 120.$$

The curves c_{18}, c_{60} touch in t points T, and intersect in the m points M. Hence they have $2t + m = 360$ common points.

12. **Web with one basis point.** Let $A = (0, 0, 0, 1)$ be on all the quadrics of the web. The equations of transformation may be expressed in the form

$$\rho x'_i = x_4 x_i + \phi_i \qquad (i = 1, 2, 3),$$

$$\rho x'_4 = \phi_4,$$

wherein ϕ_i are homogeneous and quadratic in x_1, x_2, x_3. The point A is a fundamental point; its image is the plane $x'_4 = 0$. Not all the images of a point in $x'_4 = 0$ are at A; the residual locus is the cone $\phi_4 = 0$ having A for vertex.

Every line through A joins two associated points, hence each has a line in (x') for image. The image of the line $x_i = \lambda_i r$ $(i = 1, 2, 3)$ is the line

$$x'_i = x_4 \lambda_i + \phi_i(\lambda) r, \qquad x'_4 = \phi_4(\lambda) r.$$

The points in which these lines in (x') meet the singular plane $x'_4 = 0$ are projective with the lines of the bundle at A. To the directions of the generators of the cone $\phi_4 = 0$ correspond the points of a singular conic $\phi'_4 = 0$ in $x'_4 = 0$. The cone $\phi_4 = 0$ is the tangent cone to K_4 at A, which is a double point on the surface.

Two general quadrics of the web intersect in a quartic curve which meets K_4 in 14 points apart from A. Hence L' is of order 14.

A quadric of the web meets K_4 in a curve of order 8 and genus 8, hence a plane section of L'_{14} is of genus 8. The plane $x'_4 = 0$ touches L'_{14} along the conic $\phi'_4 = 0, x'_4 = 0$. The surface K_4 has an equation of the form

$$x_4^2 \phi_4 + x_4 \psi + f = 0.$$

Each generator of $\phi_4 = 0$ meets K_4 in one point apart from A, on the surface $x_4 \psi + f = 0$. The curve of intersection is of order 8 and has a six-fold point at A. The image is the curve

$$x'_i = \phi_i \psi - f x_i, \qquad i = 1, 2, 3, \qquad x'_4 = 0,$$

in which the parameters x_1, x_2, x_3 are connected by the relation $\phi_4 = 0$. This curve is rational and of order 10; it is the residual intersection of $x'_4 = 0$ with L'_{14}. The images of the lines of $\phi_4 = 0$ are tangents to this curve c'_{10}.

Since L' is of order 14, it follows that the order of R is 20. A straight line meets L'_{14} in 14 points; its image c_4 meets R_{20} in 70 points not at A. Since c_4 has a simple point at A, it follows that R_{20} has a 10-fold point at A.

A general line l through A meets K_4 in two residual points P_1, P_2. Its image is a line l' tangent to L'_{14} at P'_1, P'_2 and meeting L'_{14} in ten other points, images of the residual intersections of l with R_{20}.

Any point P' in (x') has 8 images in (x), one of which is A; through A can be drawn seven lines of the bundle to the other associated points, hence the image congruence in (x') is of order 7. Since any quadric of the web contains two lines through A, the image congruence is of class 2. Since the points of $x'_4 = 0$ are projective with the lines through A, it follows that no two lines not in $x'_4 = 0$ can meet in a point of $x'_4 = 0$. If P' is chosen in $x'_4 = 0$, it follows that of the seven lines of the congruence through P', six must be in $x'_4 = 0$, and be tangent to c'_{10}. Hence c'_{10} is of class 6; it must therefore have 24 double points and 12 cusps.

The image of l' is l and a residual c_3 meeting K_2 (the cone $\phi_4 = 0$) in 6 points, but not passing through A. The six generators of K_2 through (c_3, K_2) have for images the 6 tangents to c'_{10} through P'. These 6 points and the direction of l at A are the 7 images of P', apart from A itself.

The lines of A, projective with the points of c'_{10}, are the generators of the tangent cone K_{10} to R_{20} at A. This cone has 24 double generators and 12 cuspidal generators.

A plane section of L'_{14} has 23 bitangents, those accounted for in the general case and the section with $x'_4 = 0$.

The cuspidal curve of L'_{14} is therefore of order 30, and the double curve of order 40. The contact curve of R_{20} and K_4 is still of order 18, and has a

six-fold point at A. The curve of intersection is of order 44 and has an 8-fold point at A. The two tangent cones K_2, K_{10} touch each other along the six tangents to the contact curve, and intersect in the eight tangents to the curve of intersection.

The curve c_{10} and the conic $\phi_4' = 0$ touch at the six image points in $x_4' = 0$, and intersect at the eight image points. The cuspidal curve c_{30}' has second order contact with $\phi_4' = 0$ at the six points, while the double curve c_{40}' has simple contact with $\phi_4' = 0$ at the eight points.

The complete image of c_{30}' is the contact curve c_{18} counted three times, and a residual cuspidal curve c_{66} on R_{20}, having a 12-fold point at A; the tangents at A being the images of the cusps of c_{10}'. The complete image of c_{40} is the curve of intersection c_{44} counted twice, and a residual double curve c_{72} on R_{20}, having a 24-fold point at A, the tangents being the images of the double points of c_{10}'.

Hence we have

$$2t + m = 264 - 24 = 240,$$

$$3p + m = 288 - 48 = 240,$$

and from Zeuthen's formula

$$3p = 2t.$$

The complete image of c_{10}' is c_8 counted twice and a residual c_{24} having an 8-fold point at A. The complete image of $\phi_4' = 0, x_4' = 0$ is A and a curve of order 8 having a six-fold point at A. The surfaces R_{20}, $\phi_4 = 0$ touch along this curve and intersect in c_{24}.

13. **Curves on composite quadrics.** Given a line γ on K_4, let π_1 pass through A. The image of γ is γ' in π'. Of the eight images in (x) of a point on π', four are in π_2, and four in π_1, of which one is fixed at A. The complete image of γ' is γ counted twice, a nodal cubic g_1 in π_1, with node at A, and a cubic g_2 in π_2. Both planes touch R_{20} along these cubics. The plane π_1 meets K_4 in a cubic $c_{3,1}$ having a double point at A. Of the nine points $(g_1, c_{3,1})$, three are on γ, four are at A, and two others. The image of $c_{3,1}$ is c_4' and of $c_{3,2}$ is c_6' in π'; together they constitute the residual section of π' with L_{14}'. The curve c_4' touches γ' in three points, images of $(\gamma, c_{3,1})$, and cuts it in two points, images of $(g_1, c_{3,1})$ not at A nor on γ. The complete image of c_6' is $c_{3,2}$ counted twice, a residual c_6 in π_2, and a c_{12} in π_1, with a six-fold point at A. The complete image of c_4' is $c_{3,1}$ counted twice, a conic in π_1 not passing through A, and a c_8 in π_2.

The plane π_1 meets the contact curve c_{18} in 9 points on γ, six points at A, and three other points on $c_{3,1}$. Hence c_4' has three cusps.

The plane π' meets the nodal curve c_{40}' in the six points of intersection (c_6', γ') each counted twice, the two points of intersection (c_4', γ') each counted twice, and in the 24 intersections (c_4', c_6').

14. **Web with two basis points.** Let $A \equiv (0, 0, 0, 1)$, $B \equiv (0, 0, 1, 0)$ be the two basis points. The equations of transformation may be expressed in the form

$$\begin{aligned} x_1' &= x_4 x_3 + a_1 x_3 x_1 + \phi_1(x_1, x_2), \\ x_2' &= x_4 x_2 + a_2 x_3 x_1 + \phi_2(x_1, x_2), \\ x_3' &= x_4 x_1 + \quad + \phi_3(x_1, x_2), \\ x_4' &= \quad a_4 x_3 x_2 + \phi_4(x_1, x_2). \end{aligned}$$

The image of the point A is the plane $x_4' = 0$, and of the point B is the plane $x_3' = 0$. Any point of the line AB has for image the single point $(AB)' \equiv (1, 0, 0, 0)$. The complete image in (x) of $(AB)'$ is the line AB and four points not on the line. The image of the line $x_3' = 0, x_4' = 0$ consists of the line AB and the cubic curve $x_4 x_1 + \phi_3 = 0$, $a_4 x_3 x_2 + \phi_4 = 0$ which meets AB in two points. The four residual points, images of $(AB)'$, are the intersections of the cubic and the planes

$$a_4 x_2^2 \phi_3 + a_2 x_1^2 \phi_4 - a_4 x_1 x_2 \phi_2 = 0 \text{ through } AB.$$

The points A, B are both double points on K_4; the cone $a_3 x_3 x_2 + \phi_4 = 0$ is tangent cone to K_4 at A, and $x_4 x_1 + \phi_3 = 0$ is tangent cone to K_4 at B. The line AB lies entirely on K_4.

The surface L' is of order 12; the planes $x_3' = 0, x_4' = 0$ are both singular, touching it along conics. The point $(AB)'$ lies on both conics and is a double point on L'. The residual surface R is of order 16; it contains the residual cubic, image of $x_3' = 0, x_4' = 0$.

The image of a line l through A is a line l' meeting $x_4' = 0$ at Q' and $x_3' = 0$ at T'. The point Q' is the image of the direction of l at A; T' is the image of the direction of the cubic curve, residual to l, at B. The line l' is bitangent to L_{12}' and meets it in 8 other points. Hence l meets R_{16} in 8 points apart from A, so that A is an 8-fold point on R_{16}. Similarly, B is 8-fold on R_{16}. The line AB does not lie on R_{16}.

The residual cubic of l meets the cone $a_4 x_3 x_2 + \phi_4 = 0$ at B and at five other points. The five generators of the cone which pass through these points have for images five tangents to the curve c_8', the residual section of L_{12}' by $x_4' = 0$, apart from the conic of contact. The curve c_8' is of class 5 and is rational, hence it has 12 double points and 9 cusps. The tangent cone K_8 to R_{16} at A has therefore 12 double generators and 9 cuspidal generators.

A general quadric of the web meets K_4 in a curve of genus 7, hence a plane section of L_{12}' is of genus 7; since it has 24 bitangents, we conclude that the double curve of L_{12}' is of order 24, and the cuspidal curve is of order 24. The curve c_8' touches the singular conic of its own plane in 5 points, and intersects

it in 6. The contact curve of R_{16}, K_4 is of order 17, and has a 5-fold point at A and at B. The curve of intersection is of order 30 and has a 6-fold point at A and at B. The tangent cones to K_4 and R_{16} at A touch along the generators which are images of the points of contact of c_8' and the singular conic in $x_4' = 0$; they intersect along the generators which are images of the six points of intersection. The cuspidal curve on R_{16} is of order 45, with A and B as 9-fold points; the double curve is of order 36, having A, B each 12-fold.

We now have the relations

$$2t + m = 180 - 36 = 144,$$

$$3p + m = 144 - 48 = 96,$$

$$3p = t,$$

hence $t = m = 48$, $p = 16$.

The image of c_8' is the c_7, intersection of K_4 and the tangent cone at A, counted twice, and a c_{18} having A, B each 6-fold. The image of the conic in $x_4' = 0$ is AB, and another c_7 with A 5-fold, B simple. The curve c_{18} and this c_7 counted twice is the complete intersection of R_{16} and $a_4 x_3 x_2 + \phi_4 = 0$.

15. **Two basis points; composite quadrics.** The lines γ are now of two different types. In the first type γ_1 one plane π_1 contains both A and B. In the second type γ_2 one plane π contains A, the other B.

The image of γ_1 is γ_1' in π_1'. The images of a point in π_1' lie four in π_2, and two in π_1, besides A and B. The complete image of γ_1' is γ_1 counted twice, the line AB, a conic g_1 through A and B, and a cubic g_2 in π_2. The plane π_1 meets K_4 in γ_1, AB, and a conic c_2 through A and B. The conics g_1, c_2 intersect on γ_1. The image of c_2 is a conic c_2' having contact with γ_1' at the images of (g_1, c_2). The plane π_2 meets K_4 in a cubic c_3 whose image in π_1' is c_6', which touches γ_1' in three points and intersects it in six.

The complete image of c_2' consists of c_2 counted twice, no other locus in π_1, and a c_4 in π_2. The complete image of c_6' is c_3 counted twice, a c_6 in π_2, and a c_{12} in π_1 having a six-fold point at A and at B. The plane π_1 meets the contact curve c_{17} in 7 points on γ, 5 at A, and 5 at B. The plane π_1' meets the cuspidal curve c_{24}' in the two points of contact (γ_1', c_2') and the three points of contact (γ_1', c_6'), each counted three times, and in the nine cusps of c_6'. The nodal curve c_{24}' meets π_1' in the six intersections (γ_1', c_6') each counted twice, and in the twelve intersections (c_2', c_6'). The plane π_2 meets the contact curve c_{17} in 8 points on γ_1 and nine points on c_3.

In case of a line γ_2, the two component planes enter symmetrically. Of the variable images of a point in π_2', three are in each plane. The complete image of γ_2' consists of γ_2 counted twice, and a nodal cubic g in each plane π.

The plane π meets K_4 in a nodal cubic c_3, meeting g in (γ_2, c_3) and in two other points not at the double point. The image of c_3 is c_4', which touches γ_2' in three points, images of (γ_2, c_3), and intersects it in two points, images of (g, c_3). The complete image of c_4' is c_3 counted twice, a c_7 in π_2 with a double point at B, and a c_3 in π_1 with a double point at A. The contact curve c_{17} meets π in 9 points on γ_2, 5 points at A, and in three points on c_3. Hence c_4' has 3 cusps.

16. **Three basis points.** Let A, B, C be three points common to all the quadrics of the web. Then A, B, C are all double points on K_4, and the lines AB, BC, CA lie on the surface, besides the lines γ_i. These three lines go into double points C', A', B' on L', and the vertices go into conics of contact.

One line γ_1 lies in the plane ABC, and one component plane of the composite quadric to which it belongs is the plane ABC. The image of this composite quadric is the plane $A'B'C'$.

The bundle A, B, C have congruences of bitangents of L' for images, each of order 5 and class 2.

L' is now of order 10, and any plane section is of genus 6, since it is the image of a (4, 4) curve on a quadric, but having three double points. The cuspidal curve on L_{10}' is of order 18; the double curve is of order 12.

The surface R is of order 12, and has A, B, C for six-fold points. On L_{10}' the cuspidal curve is of order 18, and the double curve of order 12. The contact curve of K_4, R_{12} is of order 15, and has $3P_4$. The curve of intersection is of order 18 and has $3P_4$. The cuspidal curve on R_{12} is of order 27 and has $3P_6$. The double curve on R_{12} is of order 12 and has $3P_4$. The tangent cones A, B, C have 4 tangents with 4 point contact with K_4. In α', the residual curve to the singular conic is of order 6, class 4, and genus 0; it has 6 cusps and 4 double points.

This curve meets the conic in 4 contacts, each counting for 3 cusps, and 4 intersections, each counting as 2 double points.

We now have

$$2t + m = 108 - 36 = 72,$$

$$m = 48 - 24 = 24,$$

$$p = 0.$$

Hence

$$t = m = 24.$$

The lines γ are of two kinds; γ_1 lies in the plane ABC, and the other component plane passes through no basis point. The image γ_1' passes through the three double points of L_{10}'. The complete image of γ_1' consists of γ_1 counted twice, the three lines AB, BC, CA, and a cubic g_3 in π_2.

The plane ABC of K_4 has no residual curve; the plane π_1 meets K_4 in a cubic c_3 whose image in π_1' is a c_6'. The contact curve c_{15} has 9 points on c_3, hence c_6' has 9 cusps. It touches γ_1' in three points and meets it in six.

The composite quadrics through the remaining lines γ_2 consist of a plane π_1 through two basis points, and a plane π_2 through the other, hence γ_2 meets but one line as AB. The image conic γ_2' passes through one double point. The complete image in (x) of γ_2' consists of γ_2 counted twice, the line AB, a conic g_2 in π_1 through A and B, and a cubic g_3 in π_2, having a double point at C. The plane π_1 meets K_4 in a conic c_2 through A and B, and meeting g_2 on γ_2; the plane π_2 meets K_4 in a cubic c_3 having a double point at C. The curves (g_3, c_2) meet in two points apart from γ_2 and C. The image of c_2 is c_2' having double contact with γ_2'; the image of c_3 is c_4' touching γ_2' in 3 points.

The plane π_1 meets the contact curve c_{15} in 7 points on γ_2 and 8 points at A and B; the plane π_2 meets it in 8 points on γ_2, 4 points at C, and 3 points on c_3. Hence c_4' has 3 cusps.

17. **Four basis points.** Let A, B, C, D, the vertices of a proper tetrahedron, be fixed basis points. K_4 now has four double points and six additional lines. The surface L' is of order 8, has 14 singular planes, and 6 nodes. A plane section is of genus 5. The cuspidal curve is of order 12, and the double curve is of order 4.

The bundles of lines through A, B, C, D have for images congruences of order 4, class 2, having L_8' for complete focal surface.

The surface R is of order 8, and has a 4-fold point at each basis point. The contact curve is of order 12 and has 4 three-fold points; the curve of intersection is of order 8, and has double points at the basis points.

We now have

$$p = 0, \qquad m = 0, \qquad t = 12.$$

The lines γ are again of two kinds; in one case, π_1 contains three points, π_2 one; in the other, each contains two. The image plane of the former contains a c_4' with three cusps; in the latter, we have two conics, each having double contact with γ'. R_8 has a cuspidal curve of order 12, having a triple point at each basis point. It has no double curve.

18. **Five basis points.** The surface K_4 now has five double points and ten additional lines. The surface L' is of order 6, has fifteen singular planes, and ten double points. It has a cuspidal curve of order 6, and no double curve. The surface R is of order 4 and has the basis points for nodes. It has no cuspidal or double curve. The curve of contact is of order 8 and has double points at the basis points. There is no curve of intersection. K_4, R_4, L_6' are all in $(1, 1)$ correspondence.

In this case

$$p = m = t = 0.$$

The lines γ are all of the same kind; one plane passes through three basis points, the other through two. In every singular plane the residual section is a conic having double contact with the singular conic.

The images of the 5 bundles are congruences of order 3, class 2, having L_6' for complete focal surface.

19. **Web with a basis line.** Most of the properties of this and other special cases have already been developed.*

Let d be the common basis line. It is a three-fold line on K_4, which is now a ruled surface. Every coincidence not on d belongs to a composite quadric. The images of the points of d are generators of a quadric F_2' in (x'). L' is a developable surface of order 6; the cuspidal curve is of order 6, and the double curve of order 4. The image of L_6' consists of K_4 taken twice and a developable quartic having a cuspidal cubic and no double curve. The image of the cuspidal curve c_6' consists of a curve of contact c_5' of order 5 counted three times, meeting d in four points, and the cuspidal curve of R_4. The image of the double curve c_4' is the curve of intersection of K_4, R_4. It is of order 6 and meets d in the points (d, c_5'). The configurations in the cases having additional basis elements can now be obtained readily. Similarly for other special webs.†

2. Transformations of K_4

20. **Systems of curves on K_4.** The surface K_4 is a particular case of those having an equation of the form $|a_1 b_2 c_3 d_4| = 0$, in which the elements are general linear functions of (x).‡ The general surface contains two triply infinite systems of sextics of genus 3, C_6 and $\overline{C}_6$, each cut from the surface by the cubics through a curve of the other. The system of plane sections $|C_4|$ and either system of sextics constitute a minimum base on the surface. In the case of K_4 it is necessary to enlarge the base on account of the lines γ_i. We shall now prove the following theorem.

Theorem. *The basis number of K_4 is* 11. *The systems* $|C_4|$, $|C_6|$ and *nine lines γ_i constitute a minimum base.*

Consider the transformation defined by $x_i' = D_i(x)$, D_i being the cofactor of d_i in the determinant $|a_1 b_2 c_3 d_4|$. It is involutorial and under it

$$C_4 \sim C_6, \qquad C_6 \sim C_4.$$

The image of any line γ_i is a space cubic (γ_i). Since $[C_4, (\gamma_i)] = 3$, hence γ_i is a trisecant of every curve of the system C_6. Any cubic F_3 of the system

* See *Encyklopädie der mathematischen Wissenschaften*, III, C, 2, no. 143 for the literature.

† In particular see *Encyklopädie*, l. c., no. 142.

‡ Snyder and Sharpe, *Certain quartic surfaces belonging to infinite discontinuous cremonian groups*, these Transactions, vol. 16 (1915), pp. 62–70.

D which meets K_4 in (γ_i) also cuts K_4 in another cubic. Since $C_4 \sim C_6$, the residual $C_4 - \gamma_i$ is transformed into a cubic of genus 1, hence a plane cubic.

Since $[\gamma_i, C_4] = 1$, it follows that $[(\gamma_i), C_6] = 1$. But $[\gamma_i, C_4 - \gamma_i] = 3$, hence $[(\gamma_i), (C_4 - \gamma_i)] = 3$, and $[(C_4 - \gamma_i)]$ meets γ_i in three points. A plane cubic lying in a plane through γ_i is therefore transformed into a plane cubic whose plane passes through γ_i. It follows further that $[\gamma_i, (\gamma_i)] = 0$, but $[\gamma_i, (\gamma_k)] = 2$.

The surface of trisecants of C_6 is of order 8 and genus 3, having C_6 for triple curve. The intersection of this ruled surface with K_4 consists of the ten lines γ_i and a curve of order 4 and genus 3, hence a plane curve. Thus,

$$8C_4 \equiv 3C_6 + \sum\gamma_i + C_4 \quad \text{or} \quad 7C_4 \equiv 3C_6 + \sum\gamma_i.$$

The 12 systems $|C_4|$, $|C_6|$, γ_i are therefore not independent.

But if we form the determinant of eleven systems, omitting γ_1, it is found to be different from zero. Hence these eleven are independent. It is convenient, however, to retain all 12 systems which satisfy the preceding identity.

21. **Transformation of conjugate points on K_4.** When we can express the images of the curves of the base under any transformation, the image of any other system is determined. The transformation $x_i' = D_i$ is that of conjugate points as to the web. It is defined by the equations

$$S \qquad \begin{aligned} C_4 &\sim C_6, \\ C_6 &\sim C_4, \\ \gamma_i &\sim C_6 - C_4 + \gamma_i \equiv (\gamma_i). \end{aligned}$$

This transformation will be denoted by S.

22. **Transformation by bisecants of $C_6 - C_4 + \gamma_i$.** The bisecant of the space cubic (γ_i) through P on K_4 meets K_4 in the residual point P'. The transformation $P \sim P'$ is involutorial. The bisecants from points of a plane section C_4 describe a ruled surface R_{10} of order ten on which (γ_i) is five-fold. The residual intersection with K_4 is the image of C_4, apart from the nine lines $\gamma_k (k \neq i)$. It has the symbol $10C_4 - C_4 - 5(C_6 - C_4 + \gamma_i) - (\sum\gamma - \gamma_i)$. Hence

$$C_4 \sim 7C_4 - 2C_6 - 4\gamma_i.$$

The bisecants of (γ_i) from points on γ_i describe a ruled surface R_4 of order 4, having (γ_i) for double curve. Since no other γ_k intersects γ_i, γ_k can not lie on the ruled surface. The complete intersection of R_4 and K_4 consists of γ_i, (γ_i) counted twice, and a residual curve of order 9, the proper image of γ_i. Hence

$$\gamma_i \sim 6C_4 - 2C_6 - 3\gamma_i.$$

The lines γ_k are all bisecants of (γ_i), hence remain invariant. The image of C_6 could be obtained in the same way, but is found more easily from the fact that the identity must remain invariant. We find

$$C_6 \sim 12C_4 - 3C_6 - 8\gamma_i.$$

Collecting these results, we have

T
$$\begin{aligned} C_4 &\sim 7C_4 - 2C_6 - 4\gamma_i, \\ C_6 &\sim 12C_4 - 3C_6 - 8\gamma_i, \\ \gamma_i &\sim 6C_4 - 2C_6 - 3\gamma_i, \\ \gamma_k &\sim \gamma_k. \end{aligned}$$

This transformation will be designated by T. There are ten different transformations of type T.

23. **Transformations by transversals of γ_i, γ_k.** This transformation is also involutorial. The transversals which meet C_4 form an R_6 having γ_i, γ_k for triple lines, hence

$$C_4 \sim 5C_4 - 3\gamma_i - 3\gamma_k.$$

Each curve of $|C_4 - \gamma_i|$, $|C_4 - \gamma_k|$ is invariant, hence

$$\begin{aligned} \gamma_i &\sim 4C_4 - 2\gamma_i - 3\gamma_k, \\ \gamma_k &\sim 4C_4 - 3\gamma_i - 2\gamma_k. \end{aligned}$$

The transversals of γ_i, γ_k and any other line γ_n describe a quadric surface, so that

$$\gamma_n \sim 2C_4 - \gamma_i - \gamma_k - \gamma_n.$$

Using the same method as before for C_6 we obtain

$$C_6 \sim 6C_4 - C_6 - 3\gamma_i - 3\gamma_k.$$

There are 45 transformations of this type; they will be designated by U.

24. **Transformations by secants of γ_i, (γ_k).** This is also involutorial. Using the same methods as before, we find for this transformation

V
$$\begin{aligned} C_4 &\sim 12C_4 - 3C_6 - 3\gamma_i - 7\gamma_k, \\ C_6 &\sim 17C_4 - 4C_6 - 3\gamma_i - 11\gamma_k, \\ \gamma_i &\sim 5C_4 - C_6 - 2\gamma_i - 3\gamma_k, \\ \gamma_k &\sim 11C_4 - 3C_6 - 3\gamma_i - 6\gamma_k, \\ \gamma_n &\sim 3C_4 - C_6 - \gamma_i - \gamma_k - \gamma_n. \end{aligned}$$

There are 90 of these transformations; they will be designated by V.

The product of any two of the transformations S, T, U, V is non-periodic, hence these operations generate a discontinuous group of infinite order.

25. **Transformation of K_4 into the symmetroid.** Under the transformation $x_1' = A_1$, $x_2' = B_1$, $x_3' = C_1$, $x_4' = D_1$, the equation of K_4 is transformed into $\Delta' = 0$, in which Δ' is a symmetric determinant, the elements being linear functions of x'. This is the equation of the symmetroid.*

The fundamental sextic in (x) belongs to the system C_6. The curves of the other system $\overline{C}_6$ are transformed into plane sections C_4' of the symmetroid. The plane sections C_4 of K_4 are transformed into a system of sextics C_6' on the symmetroid, passing through the ten double points P_i' images of the lines γ_i. On account of the symmetry, there is only one such system of sextics. They are cut out in pairs by the cubic surfaces through the ten double points P', which are the images of the planes in (x).

Hence

$$2C_6' = 3C_4' - \sum P_i';$$

this is the transform of the identity in (x). The relations between the two systems are now

$$C_4 \sim C_6', \qquad C_6 \sim 3C_6' - C_4', \qquad \gamma_i \sim P_i'.$$

26. **The general quartic surface through a sextic curve of genus 3.** The transformation $x_i' = D_i$ sends $F = |a_1\, b_2\, c_3\, d_4| = 0$ into a similar quartic $F' = 0$, the transformation $x_1' = A_1$, $x_2' = B_1$, $x_3' = C_1$, $x_4' = D_1$ sends $F = 0$ into a similar quartic $F'' = 0$. By a transformation of the same kind $F' = 0$ can be transformed into $F = 0$ or into $F'' = 0$, and $F'' = 0$ into $F = 0$ or into $F' = 0$. Hence we may transform $F = 0$ into $F' = 0$, then $F' = 0$ into $F'' = 0$, and finally $F'' = 0$ into $F = 0$. The series of transformations may be expressed as follows:

$$T_1 \begin{matrix} C_4 \sim C_6', \\ C_6 \sim C_4', \end{matrix} \qquad T_2 \begin{matrix} C_4' \sim 3C_4'' - C_6'', \\ C_6' \sim 8C_4'' - 3C_6'', \end{matrix} \qquad T_3 \begin{matrix} C_4'' \sim 3C_4 - C_6, \\ C_6'' \sim C_4. \end{matrix}$$

Hence the transformation

$$T_1\, T_2\, T_3 \begin{matrix} C_4 \sim 21C_4 - 8C_6, \\ C_6 \sim 8C_4 - 3C_6, \end{matrix}$$

leaves K invariant. It is not periodic.†

* Cayley, *Collected Works*, vol. 7, pp. 133–181. See p. 134.

† See Snyder and Sharpe, loc. cit., p. 65. It is readily verified that $T_1\, T_2\, T_3$ is identical with τ^3. The transformation τ^2 is non-existent. That the transformation $F \sim F'$, etc., lead to a non-periodic transformation which leaves $F = 0$ invariant is mentioned without proof by Cayley, loc. cit., p. 159.

CORNELL UNIVERSITY

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ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO BINARY FORMS, AND OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY

J. L. WALSH

INTRODUCTION

Professor Bôcher has shown how the roots of certain algebraic invariants can be determined as the positions of equilibrium in the field of force due to properly situated repelling and attracting particles.† He considers a number of fixed particles either in a plane or on the surface of a sphere (the stereographic projection of the plane) and each of these particles is supposed to repel with a force equal to its mass divided by the distance. If a particle has negative mass, it attracts instead of repelling. The plane of the particles can be considered as the Gauss plane, and with this convention Bôcher proves the following theorem:‡

THEOREM I. *The vanishing of the jacobian of two binary forms f_1 and f_2 of degrees p_1 and p_2 respectively determines the points of equilibrium in the field of force due to p_1 particles of mass p_2 situated at the roots of f_1, and p_2 particles of mass $-p_1$ situated at the roots of f_2.*

Perhaps it is desirable briefly to indicate the proof of this theorem. We give the proof merely for the plane field. Let fixed particles of masses m_1, m_2, $\cdots$, m_n be placed at the points represented by the complex quantities e_1, e_2, $\cdots$, e_n respectively. Then at any point x of the plane, the force due to these fixed particles is in magnitude, direction, and sense

$$K\left(\frac{m_1}{x-e_1}+\frac{m_2}{x-e_2}+\cdots+\frac{m_n}{x-e_n}\right),$$

where the symbol K indicates the conjugate of the complex quantity following.

* Presented to the Society, February 23, 1918.

† Maxime Bôcher, *A problem in statics and its relation to certain algebraic invariants*, Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 469. I am indebted to Professor Bôcher for a number of suggestions concerning the present paper.

‡ L. c., p. 476.

If homogeneous variables are introduced by the formulas

$$x = \frac{x_1}{x_2}, \qquad e_i = \frac{e_i'}{e_i''},$$

the plane field described in Theorem I becomes

$$K\left[x_2\left(p_2\sum_{i=1}^{i=p_1}\frac{e_i''}{e_i''x_1 - e_i'x_2} - p_1\sum_{i=p_1+1}^{i=p_1+p_2}\frac{e_i''}{e_i''x_1 - e_i'x_2}\right)\right]$$

$$= K\left[x_2\left[p_2\frac{\dfrac{\partial f_1}{\partial x_1}}{f_1} - p_1\frac{\dfrac{\partial f_2}{\partial x_1}}{f_2}\right]\right], \tag{1}$$

where

$$f_1 = (e_1''x_1 - e_1'x_2)\cdots(e_{p_1}''x_1 - e_{p_1}'x_2),$$

$$f_2 = (e_{p_1+1}''x_1 - e_{p_1+1}'x_2)\cdots(e_{p_1+p_2}''x_1 - e_{p_1+p_2}'x_2).$$

The quantity in the brackets in (1) reduces to the quotient of the jacobian of f_1 and f_2 by f_1f_2, when Euler's theorem for homogeneous functions is applied. This completes the proof of Theorem I. It is to be noted that the jacobian vanishes not only at the points of no force, but also at the multiple roots of either form or a common root of the two forms; such a point is called a point of *pseudo-equilibrium*.

From the mechanical interpretation of Theorem I, Bôcher derives a number of results concerning the location of the roots of the jacobian with reference to the location of the roots of the ground forms.*

When we consider the mechanical system, it is intuitively obvious that there can be no position of equilibrium very near any of the fixed particles. In the Corollary to Theorem II of the present paper there is determined explicitly (and in an infinite variety of ways) a circle which can be drawn separating any one of these particles from the roots of the jacobian. If we have not one fixed particle but k particles, all attracting or all repelling, and if the remaining particles in the plane (or on the sphere) are sufficiently removed from those, then the mechanical system would lead us to expect that there could be no roots of the jacobian outside of and very near to a circle surrounding the k particles. This is a rough indication of the considerations that lead to Theorem II.

In the latter part of the paper some applications of these results are made to the roots of the derivative of a rational function.

* See, e. g., Theorem III below.

PART I

Let us consider the statical system in the plane due to fixed particles of the kind described. We shall make use of two lemmas, which, indeed, are more general than is necessary for our use.

LEMMA I. *If Q is a point exterior to the circle C whose center is O, then of all possible positions for a unit (repelling or attracting) particle on or within C, that position nearest to Q causes the particle to exert the greatest force at Q,—greatest not only in magnitude but also in component along QO. That position farthest from Q causes the particle to exert the least force at Q,—least not only in magnitude but also in component along QO.*

LEMMA II. *If Q is a point interior to the circle C whose center is O, then of all possible positions for a unit (repelling or attracting) particle on or outside of C, that position nearest to Q causes the particle to exert the greatest force at Q,—greatest not only in magnitude but also in component along QO. Of all possible positions for an attracting particle on or outside of C, that position on C which is farthest from Q causes the particle to exert the force at Q which has the greatest component in the direction and sense QO.*

The truth of each of these lemmas becomes evident upon inverting C in the circle of unit radius and center Q, noting that the force exerted at Q by a unit particle at R is in direction and magnitude $R'Q$, where R' is the inverse of R in the unit circle whose center is Q.

We shall now apply these lemmas to Theorem I. Suppose there is in the plane a circle C_1 which contains on or within its circumference k roots of f_1. Suppose there is a circle C_2—larger than C_1 and concentric with it—outside of which lie all the remaining $p_1 - k$ roots of f_1. Suppose further that there is a circle C_3—also larger than C_1 and concentric with it—outside of which lie all the roots of f_2. Then we shall try to determine a circle C_0 larger than C_1 and concentric with it and such that there is no root of the jacobian of f_1 and f_2 within the annular region between C_0 and C_1.

We denote by O the common center of C_1, C_2, and C_3, and the radii of these circles by a, b, and c, respectively. We have supposed that $a < b$, $a < c$. Set up the statical system of Theorem I and consider the force at a point Q between C_1 and the smaller of C_2 and C_3. The component in the direction and sense OQ of the force due to the k positive particles (each of mass p_2) on or within C is not less than $p_2 k/(a + r)$, where r is the distance OQ. The component in the direction and sense QO of the force due to the positive particles outside of C_2 (whose mass is $(p_1 - k)p_2$) is not greater than $(p_1 - k)p_2/(b - r)$. The component in the direction and sense QO of the negative particles outside of C_3 (whose mass is $-p_1 p_2$) is not greater than

$p_1 p_2/(c+r)$. If Q is a point of equilibrium, we must have

$$\frac{p_2 k}{a+r} \leqq \frac{(p_1-k)p_2}{b-r} + \frac{p_1 p_2}{c+r}, \tag{2}$$

$$\frac{kbc - p_1 ab - (p_1-k)ac}{(p_1-k)b + p_1 c - ka} \leqq r. \tag{3}$$

If the left-hand member of (3) is positive, we construct the circle with that radius and center O, and denote this circle by C_0. Then *it is readily seen that C_0 always lies within C_2 (unless $k = p_1$, when C_0 and C_2 coincide), and C_0 may or may not lie within C_1 and may or may not lie within C_3.* If C_0 lies outside of C_1 but within C_3, then we have shown that the annular region between C_0 and C_1 contains no point of equilibrium. This region contains no root of either form and therefore no possible point of pseudo-equilibrium. Hence the annular region contains no root of the jacobian of the forms. If on the other hand, C_0 lies outside of C_3, then between C_1 and C_3 there is no root of the jacobian of the forms.

If C_0 lies outside of C_1, it is readily shown that there are precisely $k-1$ roots of the jacobian on or within C_1. Let the k roots of f_1 that are on or within C_1 move continuously so as to coincide at the point O, while the other roots of f_1 and all the roots of f_2 remain fixed. If Q is a position of equilibrium, inequality (2) obtains whenever Q is anywhere within C_1. Hence by (3) there is no root of the jacobian within C_1 except at O; and O is a $(k-1)$-fold root. During the change of the k roots of f_1 the roots of the jacobian change continuously (at least when we refer to the sphere instead of the plane) and are never in the annular region between C_1 and the nearer of C_0 and C_3. Hence at the start there were just $k-1$ roots of the jacobian on or within C_1.

Let us determine the circle C_0 by invariant elements. Suppose a line through O cuts the circles C_i in the points C_i' and C_i'' $(i = 0, 1, 2, 3)$ where the notation is such that O separates no pair of points C_i', C_j'. We find that*

$$(C_1'', C_2', C_3'', C_0') = p_1/k.$$

Hence, for the special case that C_1, C_2, and C_3 are concentric, with C_1 in the interior of C_2 and C_3, we have proved:

THEOREM II. *Suppose that f_1 and f_2 are two binary forms, the degree of f_1 being p_1, and suppose there are k roots of f_1 which lie in a closed region T_1 bounded by a circle C_1. Suppose there is a second closed region T_2 bounded by a circle C_2,*

* We are using the following definition for the cross-ratio:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

that T_2 has no point in common with T_1, and that T_2 contains the remaining $p_1 - k$ roots of f_1. Suppose further that there is a third closed region T_3 bounded by a circle C_3 coaxial with C_1 and C_2, that T_3 has no point in common with T_1, and that T_3 contains all the roots of f_2.

1. *If the circle C_0 described below lies in the region between C_1 and C_3, then there are no roots of the jacobian of f_1 and f_2 in the region included between C_1 and C_0; furthermore, there are just $k - 1$ roots of the jacobian in T_1.*

2. *If the circle C_0 lies in the region between C_2 and C_3, then there are no roots of the jacobian of f_1 and f_2 in the region included between C_1 and C_3; moreover, there are just $k - 1$ roots of the jacobian in T_1.*

In this theorem, C_0 denotes that circle of the coaxial family to which C_1, C_2, and C_3 belong which is the locus of points C_0' such that

$$(C_1', C_2', C_3', C_0') = p_1/k.$$

*C_i' and C_j'' denote the points in which any circle T orthogonal to the circles of the family cuts the circle C_i, and the notation is such that on T, neither null circle of the family shall separate any of the pairs of points C_i', C_j' $(i, j = 0, 1, 2, 3)$.**

This theorem is proved for the case that C_1, C_2, and C_3 are not concentric by making a linear transformation that transforms them into concentric circles, with C_1 in the interior of C_2 and C_3. (Such a transformation always exists.) Since the theorem is true for this particular case, and since everything used in the theorem is invariant under linear transformation, the theorem is true as stated.†

It is also true that Theorem II refers to the sphere as well as the plane, for everything essential in the theorem is invariant under stereographic projection.

* Reference to the italicized sentence immediately below (3) will show that 1 and 2 cannot occur at the same time. It may occur that C_0 lies in neither of these positions, which in the case of concentric circles means that C_0 lies within C_1; if this is true, the theorem makes no statement about the roots of the jacobian. Also, the circle C_0 may not exist, which means that the left-hand side of (3) is negative.

If no root of f_1 lies on C_1, if no root of f_1 lies on C_2 $(k \neq p_1)$, or if no root of f_2 lies on C_3, then in case (1) no root of the jacobian can lie on C_0. This is immediately seen by omitting the equality sign in (2) and hence in (3).

Of course, a theorem similar to II can be proved for Bôcher's covariant ϕ (l. c., p. 474).

Theorem II can be applied to the roots of special types of polynomials, but as Professor Curtiss pointed out to me, the following more general theorem can be proved by means of Lemma II. This more general theorem is a special case of the theorem just suggested concerning the covariant φ.

If $f(z)$ is a polynomial of degree n all of whose roots lie outside of a circle whose center is the origin and radius b, and if k_1 and k_2 are any positive numbers, then all the roots of $k_1 z f'(z) - k_2 f(z)$ lie outside the smaller of the two circles whose common center is the origin and whose radii are b and $k_2 b / (nk_1 - k_2)$ respectively. See Laguerre, *Œuvres*, vol. I, pp. 56, 133; see also the reference to Gonggrÿp below.

† We consider the exterior of a circle, including the boundary and the point at infinity, to be a closed region.

It is readily shown that Theorem II gives in general the largest region which will be free from roots of the jacobians of all pairs of forms which satisfy the hypothesis. Let us take the circles C_1, C_2, and C_3 in their original (concentric) positions, and first suppose C_0 to lie between C_1 and C_3. Then reference to inequality (2) shows that if $k \neq p_1$, the position of the particles which determine the field of force can be chosen so that (2) becomes an *equality*, and there will be a position of equilibrium on C_0. If $k = p$, C_0 and C_2 coincide, and we can consider C_2 to coincide with C_3. In this case, or if on the other hand C_0 lies outside of C_3, there can be chosen on C_3 a multiple root of f_2, which will be a root of the jacobian.

If we take C_1 a null circle P, and if we let C_2 and C_3 coincide and denote this circle by C, we have the following result:

COROLLARY. *Suppose that f_1 and f_2 are two binary forms, the degree of f_1 being p_1, and suppose that the circle C separates P (a k-fold root of f_1) from those roots of f_2 and f_1 (other than P) which do not lie on C itself. Then the circle C_0 separates P from those roots of the jacobian of f_1 and f_2 (other than P) which do not lie on C_0 itself, where C_0 is that circle of the coaxial family determined by C and P which is the locus of points C_0' such that*

$$(P, C', C'', C_0') = p_1/k;$$

C' and C'' denote the intersections of C with the circle through P and C_0' orthogonal to C.

In the corollary, it is of course true that C_0 lies between P and C unless $k = p_1$, when C_0 and C coincide.

If we take $k = p_1$, and if C_2 and C_3 are chosen coincident, Theorem II gives the following theorem, which is due to Bôcher:*

THEOREM III. *If the roots of a binary form f_1 of degree p_1 lie in a closed region T_1 and if the roots of a second binary form f_2 of degree p_2 lie in a second closed region T_2 which has no point in common with T_1, and if these two regions are bounded by arcs of circles each one of which circles separates the interior of T_1 from the interior of T_2, then the jacobian of f_1 and f_2 has just $p_1 - 1$ roots in T_1 and $p_2 - 1$ roots in T_2.*†

* As Professor Curtiss pointed out to me, the statement given by Bôcher (l. c., p. 478) is not quite accurate. This inaccuracy has been here corrected.

† There have recently been published two results which are special cases of Theorem III, although the authors were apparently not aware of the fact.

See L. R. Ford, *On the roots of a derivative of a rational function*, Proceedings of the Edinburgh Mathematical Society, vol. 33 (1915). Several of Ford's results are generalized in the present paper.

See also B. Gonggrÿp, *Quelques théorèmes*, etc., Liouville's Journal, ser. 7, vol. 1 (1915), p. 360. Compare the former reference to Laguerre.

PART II

All the theorems concerning jacobians which were proved by Bôcher, as well as the theorems of the present paper can be immediately applied to the roots of the derivative of a rational function.

Let us take any rational function not a constant, $f(z) = u(z)/v(z)$, and suppose (as we can do with no loss of generality) that u and v have no common factor containing z. Introduce homogeneous coördinates, setting $z = z_1/z_2$, and multiply the numerator and denominator of f by z_2^n, where n is the degree of f:*

$$f(z) = \frac{z_2^n u(z_1/z_2)}{z_2^n v(z_1/z_2)} = \frac{f_1(z_1, z_2)}{f_2(z_1, z_2)}.$$

If we express $f'(z)$, the derivative of $f(z)$, in terms of J, the jacobian of f_1 and f_2, we find

$$f'(z) = \frac{J}{n}\left(\frac{z_2}{f_2}\right)^2.$$

From this relation it follows that *the roots of f' are the roots of J and a double root at infinity, except that when one of these points is also a pole of f it cannot be a root of f'.*

We shall not attempt to carry over all the results concerning the roots of the jacobian to the corresponding results for the derivative of a rational function. We merely give a few examples by way of illustration.† The following theorem is a direct application of Theorem III.

If $f(z)$ is a rational function of degree n whose roots lie in a closed region T_1 and whose poles lie in a second closed region T_2 which has no point in common with T_1, and if these two regions are bounded by arcs of circles each one of which circles separates the interior of T_1 from the interior of T_2; then all the roots of the derivative of $f(z)$ lie in T_1 and T_2, except that there are two additional roots at infinity if $f(z)$ has no pole there. Except for these two possible roots, there are just $n-1$ roots of $f'(z)$ in T_1, and if $f(z)$ has no multiple pole there are just $n-1$ roots of $f'(z)$ in T_2.‡

* The degree of a rational function is the greater of the degrees of its numerator and denominator, or the common degree if the numerator and denominator have the same degree.

† Essentially the following theorem is given by Bôcher (l. c., p. 479): "If f_1 and f_2 are two forms and if all the roots of each form either lie on a circle C or are situated in pairs of points inverse with respect to C, then all the roots of the jacobian of f_1 and f_2 also lie on C or are situated in pairs of points inverse with respect to C. On any arc of C bounded by roots of f_1 (or of f_2) and containing no root of either form there is at least one root of the jacobian."

It is true that the force at any point of C (when the statical system is set up) is in direction tangent to C. Hence, if there are *two* circles C of the theorem stated in this footnote, their intersection must be a root of the jacobian or a root of one of the ground forms.

Both of these theorems can evidently be extended to the derivative of a rational function.

‡ This is a generalization of the well-known theorem of Lucas that "the roots of the deriva-

We can immediately obtain an upper bound for the moduli of the finite roots of the derivative of a rational function. Suppose f to have m_1 finite roots (or poles) and m_2 finite poles (or roots), $m_1 > m_2$. It follows from the corollary to Theorem II that if a circle whose radius is a includes all the finite roots and poles of f, then a concentric circle of radius $a(m_1 + m_2)/(m_1 - m_2)$ includes all the finite roots of f'.

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tive of a polynomial lie within or on the boundary of the smallest convex polygon enclosing the roots of the original polynomial."

As Bôcher points out, Theorem III is also a generalization of Lucas's theorem.

The following theorem is a corollary of Theorem II: *If $f(z)$ is a polynomial of degree n which has a k-fold root at P, and if a circle whose center is P and radius a includes no root of f other than P, then the circle whose center is P and radius ak/n includes no root of $f'(z)$ other than P.*

SETS OF INDEPENDENT GENERATORS OF A SUBSTITUTION GROUP*

BY

G. A. MILLER

1. INTRODUCTION

Various theorems relating to sets of independent generators of a group of finite order were developed in a previous article published in these Transactions.† The object of the present article is to extend these theorems especially along the line of substitution groups. When the order g of such a group G is of the form p^m, p being a prime number, it is known that the number of substitutions in every possible set of independent generators of G is the same, but for every other value of g there is at least one group for which this number λ is not an invariant of G,‡ viz., the cyclic group of order g. It is clear that in this group a set of independent generators can be so chosen that λ has an arbitrary value from unity to the number of distinct prime numbers which divide g, and that λ can have no other value for such a group.

If $s_1, s_2, \cdots, s_\lambda$ is a set of independent generators of G it is easy to prove that

$$s_1^{\alpha_2} s_2 s_1^{\alpha_2'}, \cdots, s_1^{\alpha_\lambda} s_\lambda s_1^{\alpha_\lambda'},$$

where the exponents $\alpha_2, \alpha_2', \cdots, \alpha_\lambda, \alpha_\lambda'$ are arbitrary integers, is a set of $\lambda - 1$ independent generators of the group H generated by these $\lambda - 1$ substitutions. In fact, if these $\lambda - 1$ generators were not independent H could be generated by some $\lambda - 2$ of them and hence G could be generated by some $\lambda - 1$ of the substitutions $s_1, s_2, \cdots, s_\lambda$ since G is evidently generated by s_1 and H. Hence it results that when H and G are not identical then $s_1, s_1^{\alpha_2} s_2 s_1^{\alpha_2'}, \cdots, s_1^{\alpha_\lambda} s_\lambda s_1^{\alpha_\lambda'}$ is also a set of independent generators of G. It can be proved in a similar manner that

$$s^{(5)} s_3 s^{(6)}, \cdots, s^{(2\lambda-1)} s_\lambda s^{(2\lambda)}$$

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† Vol. 16 (1915), p. 399.

‡ G. A. Miller, Proceedings of the National Academy of Sciences, vol. 1 (1915), p. 7.

is a set of independent generators of the group generated by these substitutions whenever $s^{(5)}, s^{(6)}, \cdots, s^{(2\lambda-1)}, s^{(2\lambda)}$ are any substitutions of the group generated by s_1 and s_2, etc.

Little is known in regard to the range of values which λ can assume in case its value is not fixed. Some of the properties of the groups for which it can be equal to the number of the prime factors of g, which is evidently its largest possible value, were determined in the article to which we referred. In particular, it is known that all such groups are solvable.

It is interesting to note that when λ is both an invariant of the non-abelian group G and has this maximal value then g must be of the form $p^m q$, where p and q are prime numbers and $p-1$ is divisible by q. This results immediately from the fact that two operators of different prime orders contained in such a group must be non-commutative and generate a group whose order is the product of these primes. Moreover, the Sylow subgroup of order p^m must be abelian and of type $(1, 1, 1, \cdots)$, and each of its substitutions of order p must be transformed into the same power of itself by a substitution of order q. As the index of this power is the same for all the subgroups of order p contained in this Sylow subgroup there is clearly one and only one such group for every value of m and every prime divisor of $p-1$.

When G is the symmetric group of degree $n > 2$, the value of λ can clearly always be made equal to any one of the positive integers from 2 to $n-1$, but the question whether λ can ever exceed $n-1$ does not seem to have been answered. In fact, it does not seem to be known whether a transitive group of degree n having a set of more than $n-1$ independent generators exists. In the following section we shall prove that every possible substitution group of degree n having k systems of intransitivity can be generated by $n-k$ of its substitutions. In the particular case where this substitution group G is of order 2^k and has $k = n/2$ systems of intransitivity it results from the theorems noted above that every possible set of independent generators of G is composed of exactly $n-k$ substitutions, so that in this special case the general theorem just noted gives the exact number of independent generators of G.

2. Substitution groups of degree n

It is easy to verify that every substitution group of degree $n < 6$ which involves k systems of intransitivity contains at least one set of $n-k$ generating substitutions and hence it must involve a set of independent generators composed of no more than $n-k$ substitutions. If there were some substitution group for which it would be impossible to find such a set of substitutions the degree of this group would therefore exceed 5. We proceed to prove that no such group exists by proving that if there is no such group of degree less than n then there can be no such group of degree n.

When G is a transitive group $k = 1$. In this case the theorem under consideration was proved in the article noted at the beginning of § 1. When G is intransitive it can be constructed by establishing some isomorphism between two constituent groups of degrees n_1 and n_2 respectively, where $n_1 + n_2 = n$. If this is an $(\alpha, 1)$ isomorphism any set of substitutions contained in G which generates the former of these two constituents must also generate the latter and hence a set of generating substitutions of G can be so chosen that the number of the substitutions in this set does not exceed $n_1 - 1$.

If there is an (α_1, α_2), $\alpha_2 > 1$, isomorphism between the two constituents under consideration the invariant subgroup of G which corresponds to the identity of the former constituent can be generated by less than n_2 substitutions. If we add to these substitutions a set of not more than $n_1 - 1$ substitutions of G which generate the former constituent we obtain a set of no more than $n - 2$ substitutions which generate G. Hence our theorem is proved for the special case when $k = 2$.

When k exceeds 2 at least one of the two constituents of degrees n_1, n_2 must be intransitive. If we assume that this is the constituent of degree n_1 it results from the preceding proof that this constituent can be generated by a set of $n_1 - 2$ substitutions and hence G can be generated by a set of $n - 3$ substitutions whenever it contains at least three systems of intransitivity. As this process may be repeated until we arrive at transitive constituents the following theorem has been established.

THEOREM. *Every substitution group of degree n which has k transitive constituents can be generated by $n - k$ of its substitutions.*

3. SUBSTITUTION GROUPS OF ORDER p^m, p BEING A PRIME NUMBER

From the fact that the order of every transitive group is a multiple of its degree it results that a transitive group of order p^m must be of degree p^α, where the possible values of α are subject to the following condition, when β is taken as small as possible:*

$$\frac{p^\beta - 1}{p - 1} \geqq m \geqq \alpha \geqq \beta.$$

Such a transitive group G must be imprimitive, $\alpha > 1$, and as the transitive group according to which one of its systems of imprimitivity is transformed is of a lower order than G it results that G contains an intransitive subgroup H of index p, whose p transitive constituents are conjugate under G.

When H is the direct product of its transitive constituents the commutator subgroup of G is simply isomorphic with the direct product of $p - 1$ of these

* *Encyclopédie des sciences mathématiques*, tome 1, vol. 3, p. 4.

constituents and the commutator subgroup of the remaining one. As the ϕ-subgroup of G includes its commutator subgroup it results that the number of independent generators of G cannot exceed the number of the independent generators of one of these transitive constituents of H increased by one, whenever H is such a direct product. Moreover, when G is a Sylow subgroup of the symmetric group of degree p^α the number of its independent generators has clearly this maximal value. Hence the

THEOREM. *Each of the Sylow subgroups of order p^m contained in the symmetric group of degree p^α has α independent generators.*

If n is written in the form

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_1 - 1} + \cdots + a_{\alpha_1 + 1},$$

where each of the coefficients $a_1, a_2, \cdots, a_{\alpha_1+1}$ is either 0 or a positive integer less than p, the Sylow subgroup of order p^m of the symmetric group of degree n is the direct product of a_1 groups which are simply isomorphic with a Sylow subgroup of the symmetric group of degree p^{α_1}, a_2 groups which are simply isomorphic with a Sylow subgroup of the symmetric group of degree $p^{\alpha_1 - 1}$, etc.* Since the ϕ-subgroup of a direct product is the direct product of the ϕ-subgroups of the factors it results that the number of the independent generators of the Sylow subgroup of order p^m contained in the symmetric group of degree n is

$$a_1 \alpha_1 + a_2 (\alpha_1 - 1) + \cdots + a_{\alpha_1}.$$

It should be noted that a Sylow subgroup of order p^m contained in the symmetric group of degree p^α does not contain as many independent generators as some of the transitive subgroups of this Sylow subgroup, whenever $\alpha > 2$. A proof of this statement is furnished by the following method for constructing infinite systems of transitive groups of prime power orders, which have a larger number of independent generators than the Sylow subgroups in which they are contained.

Suppose that one of the transitive constituents H_1 of H has for its ϕ-subgroup invariant operators constituting an abelian group of type $(1, 1, 1, \cdots)$, and let t be a substitution of order p which transforms all the corresponding letters of the p transitive constituents of H in the same order among themselves. If $s_1 s_2 \cdots s_p$ is a substitution on all the letters of H, which is invariant under t and has the property that each of its factors belongs to a particular constituent of H, then the substitution

$$t' = s_1 s_2^2 \cdots s_{p-1}^{p-1}$$

may be supposed to be transformed by t into itself multiplied by $s_1 s_2 \cdots s_p$.

* G. A. Miller, American Journal of Mathematics, vol. 23 (1901), p. 173.

If we extend the group obtained by establishing a simple isomorphism between the transitive constituents of H by means of the group obtained by replacing s_1 in t' successively by the substitutions of the ϕ-subgroup of H_1 and then extend the group thus obtained by means of t there results a group whose ϕ-subgroup is simply isomorphic with the ϕ-subgroup of H_1 and which has therefore $n_1 + 1$ more independent generators than H_1 has, p^{n_1} being the order of the ϕ-subgroup of H_1.

Starting with the regular abelian group of order p^m and of type $(1, 1, 1, \cdots)$, and extending the intransitive group obtained by establishing a $(1, 1)$ correspondence between p such groups in such a way that each substitution is commutative with t by the simply isomorphic group obtained by establishing a $(1, 1)$ correspondence between $p - 1$ of these groups according to t' there results an H such that when it is extended by t we obtain a transitive group of degree p^{m+1} which has for its ϕ-subgroup p^m invariant operators constituting the abelian group of type $(1, 1, 1, \cdots)$ and of order p^m. Hence it results that *there is a transitive group of degree p^{m+k}, m and k being arbitrary positive integers, which has $k(m + 1)$ independent generators.*

The maximum number of independent generators in such a group of degree p^c corresponds to a maximum value of the expression $(c - m)(m + 1)$ and hence it is equal to $\frac{1}{4}(c + 1)^2$ when c is odd, and to $\frac{1}{4}(c^2 + 2c)$ when c is even. The former value corresponds to $m = \frac{1}{2}(c - 1)$, while the latter presents itself in two cases, viz., when m is either $c/2$ or $(c/2) - 1$. The ratio between the number of independent generators of a Sylow subgroup of order p^β contained in the symmetric group of degree p^c and those of certain of its transitive subgroups must therefore approach zero as c approaches infinity.

If the order of a transitive group of degree p^2 is of the form p^m the value of m must be one of the p numbers $2, 3, \cdots, p + 1$. Moreover, there is at least one such transitive group for each of these values of m. We proceed to prove that with the exception of the cyclic group each one of these possible groups has exactly two independent generators. This result follows directly from the properties of one of the largest of these groups, which is a Sylow subgroup of the symmetric group of degree p^2 and hence may be supposed to contain each of the others.

Let t be a substitution of order p which is not contained in the H of this Sylow subgroup, and let s represent a substitution of degree p contained in H. The $p - 1$ substitutions $s_1, s_2, \cdots, s_{p-1}$ which satisfy the equations

$$t^{-1} s t = s_1 s, \qquad t^{-1} s_1 t = s_2 s_1, \qquad \cdots, \qquad t^{-1} s_{p-2} t = s_{p-1} s_{p-2}$$

are clearly of degrees $2p, 3p, \cdots, p^2$ respectively, and the last of these $p - 1$ substitutions is invariant under the Sylow subgroup under consideration.

Hence t and s_{p-1} generate a transitive group of order p^2, t and s_{p-2} generate a transitive group of order p^3, $\cdots$, t and s_1 generate a transitive group of order p^{p+1}. Moreover, each of these transitive groups involves all the substitutions of the H of said Sylow subgroup which together with t do not generate a transitive group of larger order. Hence it results that *every non-cyclic transitive group of degree p^2 and of order p^m has exactly two independent generators.*

By means of this theorem it is easy to determine all the transitive substitution groups of degree p^2 whose order is of the form p^m. From the preceding paragraph it results that whenever $m > 2$ the central of such a group is of order p and that t and s' generate a group which involves no substitution of order p^2 whenever $s'_{p-1} = 1$, s' being any substitution of H and s'_1, s'_2, $\cdots$, s'_{p-1} being the successive commutators corresponding to s_1, s_2, $\cdots$, s_{p-1} of the preceding paragraph. If we use for t any substitution of order p^2 contained in the Sylow subgroup of order p^{p+1} mentioned above we obtain a second group of each of the orders p^2, p^3, $\cdots$, p^p. The number of substitutions of order p^2 in each of these groups is equal to the order of the group multiplied by $(p-1)/p$. Each of these substitutions of order p^2 generates the same subgroup of order p. Hence it results that *there are exactly two transitive groups of degree p^2 and of each of the orders p^2, p^3, $\cdots$, p^p, while there is only one such group of order p^{p+1}.* The total number of transitive groups of degree p^2 which have an order of the form p^m is therefore $2p-1$.

It is well known that the enumeration of the possible substitution groups of a given degree is a very difficult problem when this degree is large, and even for the smaller degrees it has been effected mainly by tentative methods. Hence it may be of interest to note that the formula of the preceding paragraph gives the complete enumeration of the prime power transitive groups of degree p^2, and seems to be the first formula giving a complete enumeration of all the transitive groups of a composite degree and of given orders when the number of these groups for the same degree has no upper limit. That is, the number of these groups of the same degree increases indefinitely with p.

UNIVERSITY OF ILLINOIS

THE PROBLEM OF MAYER WITH VARIABLE END POINTS*

BY

GILBERT AMES BLISS

Let y denote a set of variables $(y_1, \cdots, y_n)$ each of which is a function of x, and let $y(x_1)$, $y(x_2)$ be symbols for the end values of these functions at x_1 and x_2. The latter will also be represented when convenient by the sets $(y_{11}, \cdots, y_{n1})$, $(y_{12}, \cdots, y_{n2})$, respectively. The problem to be considered is the determination of the properties of an arc

$$(E) \qquad y = y(x) \qquad (x_1 \leqq x \leqq x_2)$$

which satisfies a system of differential equations

$$(1) \qquad \varphi_\mu(x, y, y') = 0 \qquad (\mu = 1, 2, \cdots, m < n)$$

and minimizes the first of a set of functions

$$f_\rho(x_1, y(x_1), x_2, y(x_2)) \quad (\rho = 1, 2, \cdots, r \leqq 2n+2)$$

while the rest $f_2, \cdots, f_n$ are made to vanish. It will be shown that if Ω denotes the function

$$\Omega = \lambda_1 \phi_1 + \cdots + \lambda_m \phi_m,$$

then a necessary condition for a minimum is that there shall exist m functions $\lambda_1(x), \cdots, \lambda_m(x)$ not all identically zero on $x_1 x_2$, satisfying the equations†

$$(2) \qquad \Omega_{y_k} - \frac{d}{dx}\Omega_{y_k'} = 0 \qquad (k = 1, 2, \cdots, n)$$

and making all determinants of order $r+1$ of the matrix

$$(3) \qquad \begin{Vmatrix} \dfrac{\partial f_\rho}{\partial x_1} & \dfrac{\partial f_\rho}{\partial y_{k1}} & \dfrac{\partial f_\rho}{\partial x_2} & \dfrac{\partial f_\rho}{\partial y_{k2}} \\ \Omega(x_1) - \displaystyle\sum_{k=1}^{n} \Omega_{y_k'}(x_1) y_{k1}' & \Omega_{y_k'}(x_1) & -\Omega(x_2) + \displaystyle\sum_{k=1}^{n} \Omega_{y_k'}(x_2) y_{k2}' & -\Omega_{y_k'}(x_2) \end{Vmatrix}$$

vanish. In the rows of the matrix, $k = 1, \cdots, n$, while in the columns

* Presented to the Society December 22, 1916, with the title, *A device applicable to the problems of Lagrange and Mayer in the calculus of variations.*

† The subscripts denote partial derivatives of Ω.

$\rho = 1, \cdots, r$, and the arguments in the derivatives of the functions f_ρ, and in Ω and its derivatives, are in every case those belonging to E.

The classical problem of Mayer in the calculus of variations is a special case of this when $f_1 = y_{12}$ and $f_2, \cdots, f_r$ are the functions

$$x_1 - \alpha_1, \qquad y_{j1} - \beta_{j1}, \qquad x_2 - \alpha_2, \qquad y_{k2} - \beta_{k2} \qquad (j = 1, \cdots, n\,;\ k = 2, \cdots, n),$$

the α's and β's being constants. In a recent paper Bolza* has formulated a problem in which he admits so-called finite equations not involving derivatives y' in the system (1), and for which the expression to be minimized, in the non-parametric case, is a sum

$$U = \int_{x_1}^{x_2} f(x, y, y')\,dx + G(x_1, y(x_1), x_2, y(x_2))$$

instead of the function f_1. When $G = 0$ this is the Lagrange form of problem, and when $f = 0$ it has the form stated in the first paragraph above.

It will be shown in a later section that the problem of Mayer with variable end points, stated in the first paragraph above, and the problem of Bolza are equivalent in the sense that the Lagrange multiplier rule for each is deducible from that of the other. Each of them contains all of the classical problems of the calculus of variations not involving partial derivatives as special cases, with the exception of those of a type exemplified by an example proposed by Hadamard.† But the results for the problem which is the principal study of this paper are the most symmetrical. The proofs given below are new, an important feature being the use and symmetric treatment of the hypothesis that the matrix $\| \partial\varphi_\mu/\partial y'_k \|$ $(\mu = 1, \cdots, m;\ k = 1, \cdots, n)$ is of rank m at every point of E, instead of the usual unsymmetric assumption that a particular one of the determinants of order m of the matrix is everywhere different from zero on E.

1. Preliminary notions and theorems

In the following pages it will be understood that the various indices unless otherwise expressly stated have the ranges

$$i, j, k = 1, \cdots, n;\ \mu, \nu = 1, \cdots, m;\ \rho, \sigma = 1, \cdots, r;\ \tau = m + 1, \cdots, n.$$

An equation written with one of these indices is supposed to hold for every one of its values.

* Über den "anormalen Fall" beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variabeln Endpunkten, Mathematische Annalen, vol. 74 (1913), p. 403.

† *Leçons sur le calcul des variations*, p. 176.

The arc E is supposed to have the following properties:

1. It is of class C''* and such that the functions φ_μ, f_ρ are also of class C'' near the values of their arguments on E.
2. It satisfies the equations $\varphi_\mu = f_2 = \cdots = f_n = 0$.
3. The matrix $\| \partial\varphi_\mu/\partial y_k' \|$ has rank m at every point of E.
4. The matrix

$$\left\| \frac{\partial f_\rho}{\partial x_1} \quad \frac{\partial f_\rho}{\partial y_{k1}} \quad \frac{\partial f_\rho}{\partial x_2} \quad \frac{\partial f_\rho}{\partial y_{k2}} \right\|$$

with $2n + 2$ columns and r rows is of rank r at the values of the arguments of the functions f_ρ on E.

The last two hypotheses are to secure the independence of the differential equations and of the functions f_ρ near E.

It will be shown in Section 3, as a consequence of hypothesis 3, that $n - m$ functions $\varphi_r(x, y, y')$ can always be adjoined to the m functions φ_μ in such a way that the complete set φ_i has the properties of Hypothesis 1 and a determinant $| \partial\varphi_i/\partial y_k' |$ different from zero everywhere on E. If z represents always a set $(z_1, \cdots, z_n)$ of which the first m elements are identically zero, then the arc E determines uniquely the $n - m$ remaining functions in z so that

$$\varphi_k(x, y, y') - z_k = 0.$$

This is the *extended* system of differential equations formed from the system (1).

Consider a one-parameter family of arcs

$$y_k = y_k(x, u), \qquad z_k = z_k(x, u) \qquad [x_1(u) \leqq x \leqq x_2(u)]$$

which contain E for $u = 0$ and satisfy $f_2 = \cdots = f_r = 0$ and the extended system of differential equations for every u. Its *variations* are the expressions

$$(4) \quad \xi_1 = x_{1u}(0), \quad \xi_2 = x_{2u}(0), \quad \eta_k(x) = y_{ku}(x, 0), \quad \zeta_k(x) = z_{ku}(x, 0),$$

where the subscript u denotes differentiation. It is understood that ζ as well as z always represents a set of elements $(\zeta_1, \cdots, \zeta_n)$ in which the first m are zero. The variations satisfy the *equations of variation*

$$\Phi_i(\eta) - \zeta_i = F_2(\xi, \eta) = \cdots = F_r(\xi, \eta) = 0$$

where

$$\Phi_i(\eta) = \sum_k \left(\frac{\partial\varphi_i}{\partial y_k} \eta_k + \frac{\partial\varphi_i}{\partial y_k'} \eta_k' \right),$$

$$(5) \quad F_\rho(\xi, \eta) = \frac{\partial f_\rho}{\partial x_1} \xi_1 + \sum_k \frac{\partial f_\rho}{\partial y_{k1}} (y_{k1}' \xi_1 + \eta_{k1}) + \frac{\partial f_\rho}{\partial x_2} \xi_2 + \sum_k \frac{\partial f_\rho}{\partial y_{k2}} (y_{k2}' \xi_2 + \eta_{k2}),$$

* A function is of class C'' if it has continuous first and second derivatives.

and the functions y, y' occurring explicitly and in the derivatives of φ_i and f_ρ are those defining E.

The existence theorems for linear differential equations justify the following lemma:

LEMMA 1. *The equations*

$$\Phi_i(\eta) - \zeta_i = 0 \tag{6}$$

determine uniquely a set of solutions η_i of class C'' in a neighborhood of the values $x_1 \leqq x \leqq x_2$, when functions ζ_i of class C' in this neighborhood and the initial values of the η's at a single point are given.

Consider now a system H of r sets of variations ξ_1^σ, ξ_2^σ, η_k^σ, ζ_k^σ with ζ's of class C', η's of class C'', and satisfying the equations (6). Variations of this sort with ξ_1, ξ_2 arbitrary constants are called *admissible variations.* Then with the help of the existence theorems for implicit functions and differential equations, and denoting the set $(\epsilon_1, \cdots, \epsilon_r)$ by ϵ, it is possible to prove

LEMMA 2. *For every system H of variations the equations*

$$\varphi_k(x, y, y') = z_k(x) + \epsilon_1 \zeta_k' + \cdots + \epsilon_r \zeta_k^r \tag{7}$$

determine a family of solutions $Y_k(x, \epsilon)$ with initial values

$$Y_k(x_1, \epsilon) = y_k(x_1) + \epsilon_1 \eta_k'(x_1) + \cdots + \epsilon_r \eta_k^r(x_1), \tag{8}$$

and of class C' in ϵ and C'' in x near the values

$$x_1 \leqq x \leqq x_2, \qquad \epsilon_1 = \cdots = \epsilon_r = 0$$

for which it contains the arc E. The $\eta\zeta$-variations of the family with respect to the parameter ϵ_σ are η_k^σ, ζ_k^σ.

For there is a neighborhood of the values (x, y, ϵ) satisfying the conditions

$$(x, y) \text{ on } E, \qquad \epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = 0,$$

in which the equations (7) have a set of solutions

$$y_k' = \psi_k(x, y, \epsilon)$$

of class C',* and if to these equations are adjoined those of the set $d\epsilon_\sigma/dx = 0$, then the existence theorems for differential equations establish the existence of a set of functions $Y_k(x, \epsilon)$ with the properties described in the lemma.† Since the $\eta\zeta$-variations of the family with respect to the parameter ϵ_σ satisfy the equation (6), since from (8) the initial values of the η's are the values $\eta_k^\sigma(x_1)$, and finally since the corresponding ζ-variations are clearly the functions ζ_k^σ,

* Bolza, *Vorlesungen über Variationsrechnung*, p. 166, § 22). See also Bliss, *Fundamental Existence Theorems, The Princeton Colloquium Lectures*, p. 19.

† Bolza, loc. cit., p. 168, §§ 23, 24.

it follows from Lemma 1 that the $\eta\zeta$-variations of the family with respect to ϵ_σ are the functions η_k^σ, ζ_k^σ as stated.

2. The Lagrange multiplier rule

In order to prove finally the result stated in the introduction two further lemmas will be useful.

Lemma 3. *If a system H of variations exists such that the determinant $|F_\rho(\xi^\sigma, \eta^\sigma)|$ is different from zero, then E can not make f_1 a minimum.*

For suppose such a system H of variations ξ_1^σ, ξ_2^σ, η_k^σ, ζ_k^σ and consider the functions $Y_k(x, \epsilon)$ which they define as described in Lemma 2. If

$$X_1 = x_1 + \epsilon_1 \xi_1' + \cdots + \epsilon_r \xi_1^r, \qquad X_2 = x_2 + \epsilon_1 \xi_2' + \cdots + \epsilon_r \xi_2^r,$$

$$f_\rho(\epsilon_1, \cdots, \epsilon_r) = f_\rho\big(X_1, Y(X_1, \epsilon), X_2, Y(X_2, \epsilon)\big)$$

it is readily seen that for $\epsilon_1 = \cdots = \epsilon_r = 0$ the functional determinant $|\partial f_\rho/\partial \epsilon_\sigma|$ is identical with $|F_\rho(\xi^\sigma, \eta^\sigma)|$ and is different from zero. Hence the equations

$$f_1 - u = f_2 = \cdots = f_r = 0$$

have an initial solution $u = f_1(0, 0, \cdots, 0)$, $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = 0$, at which their functional determinant with respect to $\epsilon_1, \cdots, \epsilon_r$ is different from zero. They have therefore solutions for $\epsilon_1, \cdots, \epsilon_r$ in terms of u, showing that there exist arcs in every neighborhood of E satisfying

$$\varphi_\mu = f_2 = \cdots = f_r = 0$$

and giving f_1 values u both greater and less than the value $f_1(0, \cdots, 0)$ which belongs to E. This proves the lemma.

Lemma 4. *If E makes f_1 a minimum there must be constants l_σ not all zero such that the sum*

$$l_1 F_1(\xi, \eta) + \cdots + l_r F_r(\xi, \eta) \tag{9}$$

vanishes for every set of admissible variations satisfying equations (6).

By Lemma 3 the determinant $|F_\rho(\xi^\sigma, \eta^\sigma)|$ must vanish for every system H. Let s be the maximum rank possible for the determinant and suppose H selected so that this rank is actually attained. If the constants l_σ are so chosen that the expression (9) vanishes for every variation in H then (9) must also vanish for every admissible variation whatsoever. Otherwise the maximum rank of the determinant would be greater than s.

To prove the result described in the introduction let

$$f = l_1 f_1 + \cdots + l_r f_r,$$

$$(10)\qquad \begin{aligned} p_1 &= \frac{\partial f}{\partial x_1} + \sum_k \frac{\partial f}{\partial y_{k1}} y'_{k1}, & -p_2 &= \frac{\partial f}{\partial x_2} + \sum_k \frac{\partial f}{\partial y_{k2}} y'_{k2}, \\ q_{k1} &= \frac{\partial f}{\partial y_{k1}}, & -q_{k2} &= \frac{\partial f}{\partial y_{k2}}, \end{aligned}$$

the end values of the variables in the expressions on the right being always those belonging to E. The values of p_1, p_2, q_{k1}, q_{k2} can not all be zero since in that case the matrix in Hypothesis 4 of Section 1 would necessarily be of rank less than r. From equation (5), Lemma 4, and equation (6) it follows that the equations

$$(11)\qquad p_1\,\xi_1 + \sum_k q_{k1}\,\eta_{k1} - p_2\,\xi_2 - \sum_k q_{y_k'}\,\eta_{k2} = 0,$$

$$(12)\qquad \begin{aligned} \int_{x_1}^{x_2} \sum_k \lambda_k\,\zeta_k\,dx &= \int_{x_1}^{x_2} \sum_k (\Omega_{y_k}\,\eta_k + \Omega_{y_k'}\,\eta'_k)\,dx \\ &= \int_{x_1}^{x_2} \sum_k (\Omega_{y_k} - \frac{d}{dx}\Omega_{y_k'})dx + \sum_k \Omega_{y_k'}\,\eta_k \Big|_{x_1}^{x_2} \end{aligned}$$

hold for every set of admissible variations and every set of functions $\lambda_k(x)$ of class C' on $x_1\,x_2$. But the λ's can be chosen satisfying the n differential equations with initial conditions

$$(13)\qquad \Omega_{y_k} - \frac{d}{dx}\Omega_{y_k'} = 0, \qquad \Omega_{y_k'}(x_2) = q_{k2}.$$

The sum of (11) and (12) then gives

$$\int_{x_1}^{x_2} \sum_k \lambda_k\,\zeta_k\,dx = p_1\,\xi_1 + \sum_k [\,q_{k1} - \Omega_{y_k'}(x_1)\,]\,\eta_{k1} - p_2\,\xi_2$$

as an equation which must hold for all admissible variations, and therefore by Lemma 1 for arbitrarily selected values $\xi_1, \xi_2, \eta_{k1}, \zeta_k$. It follows readily then with the help of the Fundamental Lemma of the calculus of variations* and the initial conditions (13) for the functions λ_k that

$$(14)\qquad \lambda_r = p_1 = q_{k1} - \Omega_{y_k'}(x_1) = -\,p_2 = q_{k2} - \Omega_{y_k'}(x_2) = 0.$$

It is clear therefore that the last $n - m$ functions λ_r vanish identically. But the first m do not, since the last equations would otherwise make all of the constants p_1, p_2, q_{k1}, q_{k2} vanish. The last four of equations (14) however imply that all determinants of order r of the matrix

$$(15)\qquad \left\| \begin{matrix} \dfrac{\partial f_\rho}{\partial x_1} + \sum_k \dfrac{\partial f_\rho}{\partial y_{k1}} y'_{k_1} & \dfrac{\partial f_\rho}{\partial y_{k1}} & \dfrac{\partial f_\rho}{\partial x_2} + \sum_k \dfrac{\partial f_\rho}{\partial y_{k2}} y'_{k_2} & \dfrac{\partial f_\rho}{\partial y_{k2}} \\ 0 & \Omega_{y_k'}(x_1) & 0 & -\,\Omega_{y_k'}(x_2) \end{matrix} \right\|$$

* Bolza, loc. cit., p. 25.

vanish since they are obtained by multiplying the columns of this matrix by $l_1, \cdots, l_r, -1$, adding, and equating the results to zero. The matrix (3) is obtained from the one just given by subtracting from the first and $(n+2)\,d$ columns linear combinations of the others, and by introducing in the last row the terms $\Omega(x_1), \Omega(x_2)$ which are zero. This last step is entirely artificial and unnecessary, but it turns out that the matrix (3) gives more directly the customary results for many special cases. Since the matrix (3) is obtainable as just described from the last one, it follows that (3) also must have rank less than $r+1$.

The statement just proved that when E minimizes f_1 there must exist m functions $\lambda_1, \cdots, \lambda_m$ of class C' not all identically zero on $x_1\, x_2$, satisfying the equations (2), *and making all determinants of order $r+1$ of the matrix* (3) *vanish, is equivalent to the statement that there must exist m functions λ of class C' satisfying the equations* (2), *and r constants $l_1, \cdots, l_r$ not all zero satisfying with them the equations*

$$p_1 = q_{k1} - \Omega_{y_k'}(x_1) = p_2 = q_{k2} - \Omega_{y_k'}(x_2) = 0, \tag{16}$$

where the values of p_1, q_{k1}, p_2, q_{k2} are defined by the equations (10).

For the equations (16) are the linear equations in $l_1, \cdots, l_r, -1$ whose coefficients are the columns of the matrix (15). It is further a consequence of the existence theorems for differential equations that the only solutions of the equations (2) which can vanish simultaneously are the solutions

$$\lambda_1 = \cdots = \lambda_m = 0.$$

Hence a set of solutions not identically zero must have values not all zero at x_1 and x_2, and consequently defines derivatives $\Omega_{y_k'}(x_1), \Omega_{y_k'}(x_2)$ not all zero. From the fact that the matrix (15) is of rank less than $r+1$ it follows readily therefore that multipliers $l_1, \cdots, l_r$ not all zero exist satisfying equations (16). Hence the first statement of the theorem implies the second. The proof that the second implies the first is immediate.

It is important to note that a minimizing arc E is said to be *normal* for the problem under consideration when a system H of variations can be so selected that the matrix $\| F_\rho(\xi^\sigma, \eta^\sigma) \|\ (\rho = 2, \cdots, r : \sigma = 1, \cdots, r)$ has rank $r-1$; otherwise it is *abnormal*.

For a normal minimizing arc E the constant l_1 is always different from zero, and the constants and functions $l_1 = 1, l_2, \cdots, l_r, \lambda_1, \cdots, \lambda_m$ satisfying equations (2) *and* (16) *are unique. In the abnormal case there always exists a set $l_1, \cdots, l_r, \lambda_1, \cdots, \lambda_m$ with $l_1 = 0$.*

It is clear that l_1 can not be zero in the normal case since $l_1, \cdots, l_r$ make the expression (9) vanish for every set of admissible variations, and since the system H can be selected so that the matrix

$$|| F_\rho(\xi^\sigma, \eta^\sigma) || \ (\rho = 2, \cdots, r : \sigma = 1, \cdots, r)$$

has rank $r - 1$. The constants $l_1 = 1, l_2, \cdots, l_r$ are for a similar reason seen to be unique. Furthermore two systems λ, $\bar{\lambda}$ satisfying equations (2) and (16) with these constants would satisfy

$$\Omega_{y_k'}(x_1) = \bar{\Omega}_{y_k'}(x_1).$$

Hence the differences of the corresponding elements of λ and $\bar{\lambda}$ would vanish at x_1 and satisfy equations (2). They would therefore be identically zero.

The statement of the theorem with regard to the abnormal case is an immediate consequence of the way in which the constants $l_1, \cdots, l_r$ were originally determined.

3. An auxiliary theorem

It was stated in Section 2 that it is always possible to adjoin to the functions $\varphi_\mu(x, y, y')$ a set of functions $\varphi_\tau(x, y, y')$ in such a way that the determinant $|\partial\varphi_i/\partial y_k'|$ is different from zero along the arc E. The functions φ_τ can in fact be taken in the form

$$\varphi_\tau = a_{\tau 1}(x)\, y_1' + \cdots + a_{\tau n}(x)\, y_n',$$

where the functions $a_{\tau k}$ are polynomials in x_1 and x_2. If the derivative $\partial\varphi_\mu/\partial y_k'$ formed for the arc E is denoted by $a_{\mu k}(x)$, this result is an immediate consequence of the following theorem:

Auxiliary theorem. *If a matrix $|| a_{\mu k} ||$ of continuous functions $a_{\mu k}(x)$ has rank m at every point of the interval $x_1 x_2$, then $n - m$ additional rows of polynomials $a_{\tau k}(x)$ can always be selected so that the determinant $|a_{ik}|$ is everywhere different from zero on $x_1 x_2$.*

The rank of the matrix $|| a_{\mu k} ||$ is unchanged if one row is multiplied by a continuous function of x and added to another, and a proof of the theorem for the matrix so altered implies its validity also for the original one. If the first row of $|| a_{\mu k} ||$ is multiplied by a suitable factor and added to the second, the latter can be made orthogonal to the first, and by repetitions of this process every row can be made orthogonal to every other.

Consider then the linear equations in $u_1, \cdots, u_n$ whose coefficients are the rows of an orthogonal matrix $|| a_{\mu k} ||$. If $n - m$ properly selected u's are arbitrarily chosen in a neighborhood of x_1 as continuous functions of x not vanishing simultaneously, the m remaining u's will be uniquely determined by the equations, since one at least of the determinants of the matrix is different from zero at x_1. Let ξ then be the upper bound of the values x defining intervals $x_1 x$ on which the equations have continuous solutions not vanishing

simultaneously. There is a neighborhood of ξ in which one of the determinants of the matrix remains different from zero, and x can be selected in this neighborhood between x_1 and ξ. The equations have solutions u as just described on the interval $x_1\, x$ and it is possible to extend the definitions of $n - m$ of them so that they are continuous from x_1 to $\xi + \delta$ and not simultaneously vanishing to the right of x. If the $n - m$ extended u's have been properly selected the remaining m will be uniquely determined from x to $\xi + \delta$ by the equations, showing that ξ can not be the upper bound described unless it is at x_2. Furthermore in the latter case the same argument shows that there exist n continuous solutions u not vanishing simultaneously on the interval $x_1\, x_2$.

If the row of functions u so determined is adjoined to the matrix $\| a_{\mu k} \|$ the resulting matrix has rank $r + 1$ at every point of $x_1\, x_2$. For at a point where this were not true there would exist m solutions ν of the n equations

$$u_k = \nu_1 a_{1k} + \cdots + \nu_m a_{mk}.$$

The sum of these equations multiplied by the elements of the μth row of the matrix would show that ν_μ must vanish, since the μth row is orthogonal to the u-row and all the others. The vanishing of all the ν's would, however, contradict the fact that the u's do not vanish simultaneously.

It is clear therefore that one row of continuous functions $a_{m+1,\,k}$ can be adjoined to the matrix $\| a_{\mu k} \|$ in such a way that the new matrix has rank $m + 1$. Continuation of this process leads to a determinant $| a_{ik} |$ which is nowhere zero on $x_1\, x_2$. Furthermore since every continuous function can be approximated by a polynomial with any desired degree of accuracy it follows readily that the elements of the adjoined $n - m$ rows could be chosen as polynomials if desired.

4. Relations between different types of problems

For the problem of Bolza in non-parametric form the function to be minimized is

$$U = \int_{x_1}^{x_2} f(x, y, y')\, dx + G(x_1, y(x_1), x_2, y(x_2)),$$

while the equations of condition and end conditions are

$$\varphi_\alpha(x, y, y') = 0 \qquad (\alpha = 1, \cdots, p),$$

$$\psi_\beta(x, y) = 0 \qquad (\beta = 1, \cdots, q),$$

$$\chi_\gamma(x_1, y(x_1), x_2, y(x_2)) = 0 \qquad (\gamma = 1, \cdots, r).$$

He assumes that the determinant

$$\left\| \begin{matrix} \dfrac{\partial \varphi_\alpha}{\partial y_1} & \cdots & \dfrac{\partial \varphi_\alpha}{\partial y_m} \\ \dfrac{\partial \psi_\beta}{\partial y_1} & \cdots & \dfrac{\partial \psi_\beta}{\partial y_m} \end{matrix} \right\| \qquad \begin{matrix} (\alpha = 1, \cdots, p) \\ (\beta = 1, \cdots, q) \\ (m = p + q) \end{matrix}$$

is different from zero and that the end conditions are "reduced and non-singular on E." The last means that the matrix of first derivatives

$$\psi_\beta(x_1, y(x_1)), \qquad \psi_\beta(x_2, y(x_2)), \qquad \chi_\gamma(x_1, y(x_1), x_2 y(x_2))$$

with respect to their arguments has rank $2q + r$ at the values of the arguments on E.

When $f \equiv 0$ and the equations $\psi_\beta = 0$ are absent this is exactly the form treated in the present paper, and the results which Bolza states in his Satz II* are exactly those of the next to last theorem of Section 2 above. On the other hand the problem just given is precisely that of determining an arc

$$y_0(x), y_1(x), \cdots, y_n(x) \qquad (x_1 \leqq x \leqq x_2),$$

which satisfies the differential equations

$$y_0' - f = \varphi_\alpha = \frac{\partial \psi_\beta}{\partial x} + \sum_k \frac{\partial \psi_\beta}{\partial y_k} y_k' = 0 \qquad \begin{matrix} (\alpha = 1, \cdots, p) \\ (\beta = 1, \cdots, q) \end{matrix}$$

and gives the first of the functions

$$y_{02} - y_{01} + G(x_1, y(x_1), x_2, y(x_2)), \qquad \psi_\beta(x_1, y(x_1)),$$
$$\chi_\gamma(x_1, y(x_1), x_2, y(x_2)) \qquad \begin{matrix} (\beta = 1, \cdots, q) \\ (\gamma = 1, \cdots, \gamma) \end{matrix}$$

a minimum value while making the rest vanish. It is readily seen that this has the form treated in the preceding sections, and by applying the results of Section 2 it turns out that the various constants and functions for the two problems correspond as in the following table:

(A) l_1, $l_{1+\beta} - \lambda_{p+\beta}(x_1)$, $\lambda_{p+\beta}(x_2)$, $l_{1+\beta}$, $l_{1+q+\gamma}$, $\lambda_\alpha(x)$, $-\lambda'_{p+\beta}(x)$

(B) l_0, l^0_β, l'_β, l_β, $l_{q+\gamma}$, $\lambda_\alpha(x)$, $\mu_\beta(x)$

$$\alpha = 1, \cdots, p; \qquad \beta = 1, \cdots, q; \qquad \gamma = 1, \cdots, r.$$

The second row is the one containing the elements in Bolza's notation which appear in his paper. From this table and the next to last theorem of Section 2 one deduces at once Bolza's Satz II.

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*Mathematische Annalen, loc. cit., p. 438.

SPIRAL MINIMAL SURFACES*

BY

J. K. WHITTEMORE

1. **Introduction.** The equations of any minimal surface S may be written†

$$
\begin{aligned}
x &= \frac{1}{2}\int (1-u^2)F(u)\,du + \frac{1}{2}\int (1-v^2)\Phi(v)\,dv = U_1 + V_1 \\
y &= \frac{i}{2}\int (1+u^2)F(u)\,du - \frac{i}{2}\int (1+v^2)\Phi(v)\,dv = U_2 + V_2 \qquad (1) \\
z &= \int uF(u)\,du + \int v\Phi(v)\,dv = U_3 + V_3.
\end{aligned}
$$

If the surface is real F and Φ are conjugate functions, and for a real point with a real tangent plane u and v have conjugate values.‡ The direction cosines of the normal to S are

$$
X = \frac{u+v}{uv+1}, \qquad Y = \frac{i(v-u)}{uv+1}, \qquad Z = \frac{uv-1}{uv+1}. \qquad (2)
$$

The linear element ds is given by

$$
ds^2 = (uv+1)^2 F(u)\Phi(v)\,du\,dv.
$$

The differential equations of the lines of curvature and of the asymptotic lines of S are respectively

$$
F(u)\,du^2 - \Phi(v)\,dv^2 = 0, \qquad F(u)\,du^2 + \Phi(v)\,dv^2 = 0.
$$

The curves of any surface along each of which the tangent plane to the surface makes a constant angle with a fixed direction are called the *Minding parallels* of the surface with reference to that direction; the curves of the surface along each of which the normal is parallel to a plane containing the fixed direction are called the *Minding meridians*.§ We shall consider Minding

* Read before the American Mathematical Society, December 27, 1916.

† Darboux, *Théorie des surfaces*, vol. 1, 2d ed., p. 340. These equations are due to Enneper (1864) and Weierstrass (1866).

‡ Darboux, l. c., p. 343.

§ These curves were first studied by Minding, Journal für Mathematik, vol. 44 (1852), p. 66. Darboux, l. c., p. 367.

parallels and meridians of minimal surfaces with reference to the Z axis. If in (1) we set $u = re^{\phi i}$ and $v = re^{-\phi i}$, where for real points of a real surface r is real and positive and ϕ is real, it is evident from (2) that the curves (r), that is r constant, are the Minding parallels and the curves (ϕ) the Minding meridians. In the study of spiral minimal surfaces, the subject of this paper, the curves (r) play a part analogous to that taken by the curves (ϕ) in the discussion of minimal surfaces applicable to surfaces of revolution.

If F and Φ in (1) are replaced by $e^{\alpha i}F$ and $e^{-\alpha i}\Phi$ respectively, where α is a real constant, we obtain the equations of a family of minimal surfaces called *associate* to S.* Corresponding points of associate minimal surfaces are points given by the same parameter values u, v. Schwarz proved that the locus of points on a family of associate surfaces corresponding to given values of u, v is an ellipse whose center is the origin. These ellipses we call *Schwarz ellipses*. We have found the equations of the locus L of the vertices of the Schwarz ellipses for a given minimal surface with equations (1) to be†

$$x = U_1\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_1\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}}, \qquad y = U_2\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_2\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}},$$
$$z = U_3\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_3\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}}. \tag{3}$$

In (3) the first radical of each second member has the same determination; the second is the reciprocal and conjugate of the first for real points of a real surface S. The four vertices of each ellipse are given by the four determinations of the first radical. Evidently L consists generally of four nappes symmetrical in pairs with respect to the origin.

A *spiral surface* is most simply defined as follows:‡ A spiral surface is the locus of the different positions of any curve which is rotated about an axis and at the same time subjected to a homothetic transformation with respect to a point of the axis in such a way that the tangent to the locus described by any point of the curve makes a constant angle with the axis. The spiral surfaces include surfaces of revolution but we shall hereafter in speaking of a spiral surface suppose it not to be a surface of revolution. Evidently the locus of each point of the curve generating a spiral surface lies on a right cir-

* Associate minimal surfaces were first systematically considered by H. A. Schwarz, *Miscellen aus dem Gebiete der Minimalflächen*, Journal für Mathematik, vol. 80 (1875), p. 286, see also Darboux, l. c., p. 379.

† American Journal of Mathematics, vol. 40 (1918), p. 87.

‡ Eisenhart, *Differential Geometry*, p. 151. Spiral surfaces were first studied by Maurice Lévy in a paper published in Comptes Rendus, vol. 87 (1878), p. 788. He gave them as an example of surfaces such that E, F, G, the coefficients in the square of the linear element, are homogeneous functions of the parameters of any degree other than -2. Lévy named them "surfaces pseudo-moulures logarithmiques."

cular cone whose axis is the axis of the surface, cuts all elements of the cone at the same angle, and is projected on a plane perpendicular to the axis in a logarithmic spiral. Such a curve we shall call simply a spiral.* The equations of any spiral surface may be written

$$(4) \qquad x = \rho e^v \cos(\omega + hv), \qquad y = \rho e^v \sin(\omega + hv), \qquad z = \zeta e^v,$$

where h is constant, ρ, ω, and ζ are functions of u alone, and the curves (u) are the spirals. Darboux has proved† that the orthogonal trajectories of the spirals of a spiral surface whose generating curve is given may be found by quadratures, and the square of the linear element written in the form

$$ds^2 = e^{2v} U^2 (du^2 + dv^2),$$

where U is a function of u alone. If U is given, h an arbitrary constant, and ζ, ρ, ω determined from

$$(5) \qquad h^2(\zeta^2 + \zeta'^2) = h^2 U^2 - U'^2, \qquad (h^2+1)\rho^2 = U^2 - \zeta^2,$$

$$(h^2+1)\rho^2 \omega' = -h\zeta\zeta' - \frac{1}{h} UU',$$

the surface (4) has the linear element in the form given, so that the solution of (5) determines a three-parameter family of spiral surfaces applicable to a given spiral surface.‡

Lie proved§ that all real minimal surfaces applicable to spiral surfaces are given by writing in (1)

$$(6) \qquad F(u) = cu^{m-2+ni}, \qquad \Phi(v) = \bar{c} v^{m-2-ni},$$

where c and $\bar{c}$ are conjugate constants and m and n are real constants. The number -2 is introduced in the exponents only to simplify future formulations. No essential restriction is imposed by taking the modulus of c as unity, so that we shall suppose $c = e^{\alpha i}$ where α is real. If in (6) $n = 0$ equations (1) and (6) give all minimal surfaces applicable to surfaces of revolution.‖ That these surfaces are applicable to surfaces of revolution if n is zero, and when n is not zero to spiral surfaces, appears from the linear element of (1) and (6) in terms of r, ϕ

$$ds^2 = e^{-2n\phi} r^{2m-4} (r^2+1)^2 (dr^2 + r^2 d\phi^2).$$

* These curves have been named "cylindro-conical helices." See *Encyclopädie der mathematischen Wissenschaften*, Bd. III, 3, Heft 2, 3, p. 252.

† L. c., pp. 148–150.

‡ Proved by Maurice Lévy, as stated by Darboux (l. c., p. 149). Of the three constants one is additive to ω and has no effect on the form of the surface, so that there are generally given by the integration of (5) a *two* parameter family of different surfaces.

§ Mathematische Annalen, vol. 15 (1879), p. 503. Darboux, l. c., p. 396.

‖ First proved by Schwarz, *Miscellen*, p. 296.

Evidently the Minding parallels (r) of (1) correspond to the spirals of the spiral surface, and, when n is zero, to the parallels of the surface of revolution. If $m = 0$, $n \neq 0$ equations (1) and (6) give all minimal surfaces which are spiral surfaces.*

A. Ribaucour proved† that the minimal surfaces associate to a minimal surface applicable to a surface of revolution are congruent to the given minimal surface except for $m = 0$, which value gives the minimal helicoids. Darboux states without proof that the minimal surfaces associate to a minimal surface applicable to a spiral surface are similar to the given surface. In the following section we prove these theorems, noting that exception must be made to Darboux's statement in the case of spiral minimal surfaces, and proving also a theorem, which we believe has not previously been stated, concerning the similarity to itself of the general surface given by (1) and (6). The remainder of this paper deals entirely with spiral minimal surfaces: we consider families of minimal surfaces associate to a surface of this kind, their Schwarz ellipses and the locus L of the vertices of these ellipses, the envelope of the family of associate surfaces, and the evolute surface; it is proved that on every spiral minimal surface there are an infinite number of plane spiral lines of curvature, the intersections of the surface with the plane part of L, an infinite number of spiral asymptotic lines, the intersections of the surface with the right conoids of L, and a single spiral geodesic; we show that on one member of a family of associate spiral minimal surfaces all the plane spiral lines of curvature except one are double curves; we discover certain symmetries of these surfaces, and finally obtain two characteristic properties of spiral minimal surfaces connected with the locus L and with the Minding parallels.

2. **Darboux's similarity theorem.** The first two of equations (1) may be combined and written

$$x + yi = -\int u^2 F(u)\,du + \int \Phi(v)\,dv.$$

For the minimal surfaces applicable to spiral surfaces we substitute from (6)

$$F(u) = e^{ai}\,u^{m-2+ni}, \qquad \Phi(v) = e^{-ai}\,v^{m-2-ni}, \qquad (n \neq 0).$$

When this surface is rotated about the Z axis through the angle β,

$$x' + y'\,i = e^{\beta i}(x + yi), \qquad z' = z,$$

where x', y', z' are the new coördinates of x, y, z. Sufficient conditions that this surface in the new position be similar to that surface for which

* First proved by Darboux, l. c., p. 359.

† "*Étude sur les élassoides ou surfaces à courbure moyenne nulle.*" Memoires Couronnés de l'Académie Royale de Belgique, vol. 44 (1882), chap. XX. Ribaucour disregards the cases $m = 0, \pm 1$; Darboux proves the theorem for all cases except $m = 0$ (l. c., pp. 395, 396).

$\alpha = 0$ in the original position with the origin as center, C as the ratio of similarity, and all constants of integration chosen as zero, are that u' and v' may be determined so that, with constant real values for β and C,

$$e^{\beta i}(-u'^2 F(u')\,du' + \Phi(v')\,dv') = C(-u^{m+ni}\,du + v^{m-2-ni}\,dv),$$

$$u' F(u')\,du' + v' \Phi(v')\,dv' = C(u^{m-1+ni}\,du + v^{m-1-ni}\,dv).$$

These equations are satisfied by constant real values of β and C by taking u' and v' as conjugate functions of u and v alone respectively if the same is true of the following four equations:

$$e^{(\alpha+\beta)i} u'^{m+ni}\,du' = Cu^{m+ni}\,du, \qquad e^{(\beta-\alpha)i} v'^{m-2-ni}\,dv' = Cv^{m-2-ni}\,dv,$$

$$e^{\alpha i} u'^{m-1+ni}\,du' = Cu^{m-1+ni}\,du, \qquad e^{-\alpha i} v'^{m-1-ni}\,dv' = Cv^{m-1-ni}\,dv.$$

From the last group of equations

$$u' = ue^{-\beta i}, \qquad v' = ve^{\beta i}.$$

Setting these values of u' and v' in the four equations, we have

$$C = e^{n\beta+(\alpha-m\beta)i}, \qquad C = e^{n\beta-(\alpha-m\beta)i},$$

so that we must have $\alpha - m\beta = 0$. If m is not zero the equations are satisfied by the constant real values,

$$\beta = \frac{\alpha}{m}, \qquad C = e^{n\alpha/m}. \tag{7}$$

If m is zero the proof fails. Equations (7) express the two theorems: (I) Minimal surfaces associate to a minimal surface applicable to a surface of revolution ($n = 0$) are congruent to that surface and may be made to coincide with it by a rotation about the Z axis, except in the case of the minimal helicoids ($m = 0$). That the associate minimal helicoids are not similar is well known.* (II) Minimal surfaces associate to a minimal surface applicable to a spiral surface ($n \neq 0$) are similar and are brought into similar position by a rotation about the Z axis, except in the case of the spiral minimal surfaces ($m = 0$). That the associates of a spiral minimal surface are not similar to that surface will appear in § 6 of this paper.

If we observe that the associate to any minimal surface given by $\alpha = 2k\pi$, where k is any integer, coincides with that surface, and that the associate $\alpha = (2k+1)\pi$ is symmetrical with respect to the origin to that surface, we have in (7) the proof of a theorem due to Ribaucour: A minimal surface applicable to a surface of revolution, not a minimal helicoid, is brought into coincidence with itself by rotation about the Z axis through the angle $2k\pi/m$; it appears also, though not stated by Ribaucour, that such a surface is brought

* See, for example, Scheffers, *Theorie der Flächen*, 2d ed., pp. 362, 363.

into coincidence with its symmetry with respect to the origin by rotation about the Z axis through the angle $(2k+1)\pi/m$. Finally it appears from (7) that a minimal surface applicable to a spiral surface ($m \neq 0$, $n \neq 0$) is similar to itself and to its symmetry with respect to the origin in an infinite number of positions if m is irrational, in a finite number if m is rational.

3. **Equations and linear element of $S_{n\alpha}$.** We denote by $S_{n\alpha}$ the spiral minimal surface given by substituting in (1)

$$F(u) = e^{\alpha i} u^{-2+ni}, \qquad \Phi(v) = e^{-\alpha i} v^{-2-ni}.$$

Performing the integrations indicated, taking all constants of integration as zero, then writing

$$u = re^{\phi i}, \qquad v = re^{-\phi i}, \qquad n = -\cot\beta,$$

we have the equations of $S_{n\alpha}$ in terms of the real parameters r, ϕ

$$x = e^{-n\phi}\sin\beta\left[r\sin(\phi-\beta+\alpha+n\log r) + \frac{1}{r}\sin(\phi-\beta-\alpha-n\log r)\right],$$

$$(8)\quad y = -e^{-n\phi}\sin\beta\left[r\cos(\phi-\beta+\alpha+n\log r) + \frac{1}{r}\cos(\phi-\beta-\alpha-n\log r)\right],$$

$$z = -2e^{-n\phi}\tan\beta\sin(\alpha+n\log r).$$

In these equations n is retained for the sake of abbreviation. If we let

$$n\log r = u, \qquad -n(\phi-\beta) = v, \qquad h = -\frac{1}{n}$$

and set

$$\rho\cos\omega = 2e^{-n\beta}\sin\beta\sin(u+\alpha)\sinh\frac{u}{n},$$

$$\rho\sin\omega = -2e^{-n\beta}\sin\beta\cos(u+\alpha)\cosh\frac{u}{n},$$

$$\zeta = -2e^{-n\beta}\tan\beta\sin(u+\alpha),$$

equations (8) take Darboux's form (4) thus proving that $S_{n\alpha}$ is actually a spiral surface and that the spirals are the Minding parallels (r).

The intersection of $S_{n\alpha}$ with the xy plane is given by

$$\alpha + n\log r = k\pi,$$

where k is any integer, and each such value of k gives a curve, as appears from the first two equations of (8), whose polar equation, coördinates R, Θ, is $R = e^{-n(\Theta+c)} = e^{(\Theta+c)\cot\beta}$, so that the surface cuts the plane in an infinite

number of congruent logarithmic spirals which cut all radii vectores at the angle β.

The linear element of (8) is given by

$$ds^2 = (r^2 + 1)^2 r^{-4} e^{-2n\phi} (dr^2 + r^2 d\phi^2).$$

Substituting the values u, v given after (8)

$$ds^2 = e^{2v} U^2 (du^2 + dv^2), \qquad U^2 = 4e^{-2n\beta} \tan^2 \beta \cosh^2 \frac{u}{n}.$$

We may determine the spiral minimal surfaces S_{na} applicable to a given spiral minimal surface S_0 which are given by the integration of (5) as follows. Let the subscript apply to all quantities relating to S_0; from $U^2 = U_0^2$ follows $n^2 = n_0^2$; the first of equations (5) gives

$$\zeta^2 + \zeta'^2 = 4e^{-2n\beta} \tan^2 \beta = U_0^2 - U_0'^2/h^2.$$

In this equation we may replace U_0 by U, and are led to the condition $h^2 n^2 = h^2 n_0^2 = 1$. The solutions of (5) with $h = -1/n_0$ substituted in (4) give equations (8) where n is replaced by n_0, α is the constant introduced by the integration of the equation for ζ, and the constant additive to ω, which affects only the position of the surface, is taken as zero. The integration of (5) with $h = 1/n_0$ leads to the surfaces $S_{-n_0\alpha}$, surfaces not associate to S_0 but symmetrical with respect to the X axis to the associate surfaces, as will appear in § 9. Spiral surfaces, not minimal, applicable to S_0 would be given by the substitution in (4) of the solutions of (5) for values of h not equal to $\pm 1/n_0$, but it appears to be impossible to integrate the equation for ζ for such values.

4. **The locus L for S_{na}.** From equations (1) for S_{n0} we find

$$\Sigma V_1^2 / \Sigma U_1^2 = (uv)^{-2ni};$$

that part of L given by (3) where the first radical is taken as $(uv)^{-ni/2}$, which we call L_1, has the equations, obtained by substituting this value and writing as before $u = re^{\phi i}$, $v = re^{-\phi i}$,

$$(9) \qquad \begin{aligned} x &= e^{-n\phi} \frac{r^2 + 1}{r} \sin \beta \sin (\phi - \beta), \\ y &= -e^{-n\phi} \frac{r^2 + 1}{r} \sin \beta \cos (\phi - \beta), \\ z &= 0. \end{aligned}$$

Points common to L_1 and S_{na} are given by

$$(uv)^{-ni/2} = e^{\alpha i} \qquad \text{or.} \qquad \alpha + n \log r = 2k\pi.$$

The equations of L_2, symmetrical with respect to the origin to L_1, are

obtained by changing the signs of the second members of (9). Points common to L_2 and $S_{n\alpha}$ are given by $\alpha + n \log r = (2k+1)\pi$. The curves common to L_1, L_2, and $S_{n\alpha}$ are the logarithmic spirals in which the latter cuts the xy plane.

Choosing the first radical in (3) as $i(uv)^{-ni/2}$ we obtain for L_3, part of the locus L, the equations

$$(10) \qquad x = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\cos(\phi-\beta), \qquad y = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\sin(\phi-\beta),$$
$$z = -2e^{-n\phi}\tan\beta.$$

Points common to L_3 and $S_{n\alpha}$ are given by

$$i(uv)^{-ni/2} = e^{\alpha i} \quad \text{or} \quad \alpha + n\log r = 2k\pi + \frac{\pi}{2}.$$

The equations of L_4, symmetrical to L_3, are found by reversing the signs of the second members of (10). Points common to L_4 and $S_{n\alpha}$ are given by $\alpha + n\log r = (2k+1)\pi + \pi/2$. Equations (9) and (10) may be obtained by substituting the appropriate values of r in (8) and regarding α as variable.

The surface L_3 is a spiral surface for (10) may be put in the form (4), and its spirals are the curves (r). From (10) it appears that every curve (ϕ) of L_3 is a straight line intersecting the Z axis and parallel to the xy plane, so that L_3 is a spiral right conoid.

The distances b of r, ϕ of L_1 from the origin and a of r, ϕ of L_3 from the origin are given respectively by

$$b^2 = e^{-2n\phi}\left(\frac{r^2+1}{r}\right)^2\sin^2\beta, \qquad a^2 = e^{-2n\phi}\left[\left(\frac{r^2-1}{r}\right)^2\sin^2\beta + 4\tan^2\beta\right],$$

and for all r, ϕ we have $a^2 > b^2$, so that a and b are respectively the semi-major and minor axes of Schwarz's ellipse r, ϕ; then the xy plane (L_1, L_2) is the locus of the extremities of the minor axes, and the two conoids (L_3, L_4) form the locus of the ends of the major axes. Since $a^2 - b^2$ is independent of r all ellipses corresponding to a constant ϕ have the same focal distance; the eccentricity is independent of ϕ so that all ellipses corresponding to points of a spiral (r) have the same eccentricity. The maximum eccentricity is $\sin\beta$ and is given by $r = 1$; all ellipses, $r = 1$, have their major axes on the Z axis and touch S_{n0} at the extremities of their minor axes. When r approaches zero or becomes infinite the eccentricity approaches zero, corresponding points of $S_{n\alpha}$ receding indefinitely from the origin.

5. **Surfaces related to $S_{n\alpha}$.** The *envelope* of a family of minimal surfaces associate to (1) consists of two surfaces symmetrical with respect to the origin

given by*

$$x = U_1 H + V_1 H^{-1}, \qquad y = U_2 H + V_2 H^{-1}, \qquad z = U_3 H + V_3 H^{-1},$$

where

$$H = \pm \sqrt{\frac{(u+v)V_1 + i(v-u)V_2 + (uv-1)V_3}{(u+v)U_1 + i(v-u)U_2 + (uv-1)U_3}}.$$

For E_1 and E_2, the two parts of the envelope of a family of associate spiral minimal surfaces, corresponding to the upper and lower signs respectively in H,

$$H = \pm e^{(\xi - n \log r)i}, \qquad \tan \xi = \frac{r^2 - 1}{r^2 + 1} \tan \beta.$$

The equations of E_1 are also given by writing $\xi - n \log r$ in place of α in (8). That E_1 is a spiral surface, whose spirals are the curves (r), may be shown by reducing its equations to the form (4); the expressions for $\rho \cos \omega$, $\rho \sin \omega$, and ζ for E_1 are obtained by replacing α by $\xi - u$ in the equations following (8). Points common to $S_{n\alpha}$ and E_1, $S_{n\alpha}$ and E_2 are given respectively by

$$\xi = \alpha + n \log r + 2k\pi, \qquad \xi = \alpha + n \log r + (2k+1)\pi.$$

Each of these equations has just one positive root r for every integral value of k, so that every surface $S_{n\alpha}$ is tangent to each surface of the envelope along an infinite number of spirals. The solution of the first equation for $\alpha = k = 0$ is $r = 1$, so that S_{n0} is tangent to E_1 along this curve, which will appear as a curve of particular interest.

The coördinates of the points of the two nappes of the *evolute surface* of $S_{n\alpha}$ corresponding to r, ϕ of the latter are

$$x \pm RX = x \pm e^{-n\phi} \frac{r^2 + 1}{r} \cos \phi,$$

$$y \pm RY = y \pm e^{-n\phi} \frac{r^2 + 1}{r} \sin \phi,$$

$$z \pm RZ = z \pm e^{-n\phi} \frac{r^4 - 1}{2r^2},$$

where x, y, z are the coördinates of r, ϕ on $S_{n\alpha}$ given by (8), X, Y, Z the direction cosines of the normal, and R the absolute value of either principal radius of curvature. The only interest in this evolute surface consists in the fact that it is a spiral surface, whose spirals are the curves (r), as may be seen by reducing its equations to the form (4).

In connection with the spiral surfaces related to $S_{n\alpha}$ the following considerations are of interest: the right circular cone whose axis is the Z axis

* My paper, American Journal of Mathematics, *l. c.*

and whose elements make the angle γ with this axis is

$$x^2 + y^2 = z^2 \tan^2 \gamma .$$

The spiral which cuts the elements of this cone under the angle λ is

$$x = e^{v \sin \gamma \cot \lambda} \sin \gamma \cos (c + v), \qquad y = e^{v \sin \gamma \cot \lambda} \sin \gamma \sin (c + v),$$

$$z = e^{v \sin \gamma \cot \lambda} \cos \gamma .$$

Equations (4) of any spiral surface may be written, setting $v = -n\phi$ and $h = -1/n$,

$$(4') \quad x = e^{-n\phi} \rho \cos (\omega + \phi), \qquad y = e^{-n\phi} \rho \sin (\omega + \phi), \qquad z = e^{-n\phi} \zeta .$$

The curve (r) of (4') lies on the cone $x^2 + y^2 = \rho^2 z^2/\zeta^2$, so that $\tan \gamma = \rho/\zeta$, which gives, for $r = 1$, $\tan \gamma = \cos \beta \cot \alpha$. We find

$$\tan \lambda = \sin \gamma \tan \beta ,$$

so that λ is determined by $n = -\cot \beta$ and γ. Since for fixed n (4') includes the surfaces $S_{n\alpha}$, L_3, E_1, and the evolute surface of $S_{n\alpha}$, it follows that every right circular cone whose axis is the Z axis cuts all these surfaces in congruent spirals.

6. **Special spirals on $S_{n\alpha}$.** The differential equation of the *lines of curvature* of (1) becomes for $S_{n\alpha}$

$$e^{\alpha i} u^{-2+ni} du^2 - e^{-\alpha i} v^{-2-ni} dv^2 = 0 .$$

This equation integrated gives

$$e^{\alpha i} (uv)^{ni/2} \pm 1 = cv^{ni/2} .$$

The only spirals (r) among the lines of curvature are given by $c = 0$. For them we have, belonging respectively to the two families of lines of curvature,

$$\alpha + n \log r = 2k\pi , \qquad \alpha + n \log r = (2k + 1)\pi .$$

These are the curves of intersection of $S_{n\alpha}$ with the xy plane, and are logarithmic spirals; they are moreover the curves common to $S_{n\alpha}$ and L_1, $S_{n\alpha}$ and L_2 respectively. For $\alpha = k = 0$ the first equation gives $r = 1$, which is therefore a line of curvature of S_{n0}.

The differential equation of the *asymptotic lines* of $S_{n\alpha}$ is

$$e^{\alpha i} u^{-2+ni} du^2 + e^{-\alpha i} v^{-2-ni} dv^2 = 0 ,$$

giving the integral

$$e^{\alpha i} (uv)^{ni/2} \pm i = cv^{ni/2} .$$

The only spirals among the asymptotic lines are given by $c = 0$; for these,

belonging respectively to the two families of asymptotic lines,

$$\alpha + n \log r = 2k\pi + \frac{\pi}{2}, \qquad \alpha + n \log r = (2k+1)\pi + \frac{\pi}{2}.$$

These are the curves common to $S_{n\alpha}$ and L_3, $S_{n\alpha}$ and L_4 respectively. The first of these equations gives $r = 1$ for $\alpha - \pi/2 = k = 0$. This line on $S_{n\pi/2}$ is the Z axis. Since on every surface the Minding parallels with reference to the Z axis and the curves of steepest ascent form a conjugate system,* the asymptotic spirals are curves of steepest ascent and are the only such spirals.

The properties of the Minding parallels (r) of $S_{n\alpha}$ are analogous to properties of the Minding meridians (ϕ) of minimal surfaces applicable to surfaces of revolution.† We note further that $S_{n\alpha}$ has no line of curvature or asymptotic line (ϕ) while minimal surfaces applicable to surfaces of revolution, with the exception of the catenoid and its adjoint surface, the right helicoid, have no line of curvature or asymptotic line (r).

From the linear element of $S_{n\alpha}$, given in § 3, it appears‡ that the only *spiral geodesic* of this surface is $r = 1$. This spiral being plane for $\alpha = 0$, S_{n0} has symmetry with respect to its plane, the xy plane; since, for $\alpha = \pi/2$, this spiral is the Z axis $S_{n\pi/2}$ has symmetry with respect to this axis. On any surface similar to $S_{n\alpha}$ there will correspond to the spiral geodesic, $r = 1$, a spiral geodesic on a cone whose semi-vertical angle γ is given by

$$\tan \gamma = \cos \beta \cot \alpha.$$

On the associate surface $S_{n\alpha'}$ the only spiral geodesic lies on a cone of semi-vertical angle γ' given by $\tan \gamma' = \cos \beta \cot \alpha'$. Then $S_{n\alpha}$ and $S_{n\alpha'}$ are not similar if $\alpha' - \alpha \neq k\pi$. There is no geodesic ($\phi$) of $S_{n\alpha}$.

7. **Double spirals of S_{n0}.** Writing $\alpha = 0$ in (8) the equations of S_{n0} are

$$x = e^{-n\phi} \sin \beta \left[r \sin (\phi - \beta + n \log r) + \frac{1}{r} \sin (\phi - \beta - n \log r) \right],$$

$$y = -e^{-n\phi} \sin \beta \left[r \cos (\phi - \beta + n \log r) + \frac{1}{r} \cos (\phi - \beta - n \log r) \right],$$

$$z = -2e^{-n\phi} \tan \beta \sin (n \log r).$$

If in these equations r is replaced by $1/r$, x and y are unchanged and the sign of z is reversed. If we consider those values of r satisfying the equation, $n \log r = k\pi$, we have the logarithmic spiral lines of curvature in which S_{n0}

* My paper, Annals of Mathematics, ser. 2, vol. 19, p. 4.
† My paper, Annals of Mathematics, l. c.
‡ Eisenhart, *Differential Geometry*, pp. 134, 266, 267.

intersects the xy plane; excepting the value $r = 1$, it appears that each of these spirals is given by two different reciprocal values of r. The direction cosines of the normal to the surface, given in § 1, are in terms of r, ϕ

$$X = \frac{2r\cos\phi}{r^2+1}, \qquad Y = \frac{2r\sin\phi}{r^2+1}, \qquad Z = \frac{r^2-1}{r^2+1}.$$

Changing r to $1/r$, X and Y are unaltered and the sign of Z is reversed. It follows that along each spiral line of curvature of S_{n0}, except $r = 1$, two branches of the surface intersect; moreover they intersect at a constant angle along each curve, the angle depending on r alone and approaching the limit π as r increases indefinitely from unity.

8. **Further properties of spirals on $S_{n\alpha}$.** In the first paper* published dealing with minimal surfaces applicable to surfaces of revolution E. Bour showed that every Minding meridian (ϕ) of the surface cuts at the same angle all members of either family of lines of curvature and all level curves (z). The Minding parallels (r) have the same property on $S_{n\alpha}$, as may be shown directly; we find, for example, that the angle of (r) and (z) is $\alpha + n\log r$, and is therefore constant with r.

The equation of the tangent plane to $S_{n\alpha}$ at r, ϕ is

$$x\cos\phi + y\sin\phi + z\frac{r^2-1}{2r} = P(r)e^{-n\phi},$$

$$P(r) = -\frac{r^2+1}{r}\sin^2\beta\sec\xi\cos(\alpha + n\log r - \xi),$$

where ξ is the angle introduced in § 5. The helicoidal developable tangent to $S_{n\alpha}$ along (r) therefore intersects the xy plane in the envelope of the lines, $x\cos\phi + y\sin\phi = P(r)e^{-n\phi}$, which is a logarithmic spiral congruent to the plane lines of curvature (r). It may also be shown from the equation of the tangent plane that the planes tangent to the surface along (ϕ) envelop a cylinder whose elements are parallel to the xy plane, and that the cylinders corresponding to different values of ϕ are similar and are brought into similar position by rotation about the Z-axis through the angle ϕ. Bour stated in the paper cited that in the case of a minimal surface applicable to a surface of revolution the developables tangent to the surface along a Minding meridian (ϕ) are cylinders whose elements are parallel to the xy plane; we have proved that this is a property of every non-developable surface.† Ribaucour proved in the *mémoire* cited that in the case of minimal surfaces applicable to surfaces

* "*Théorie de la déformation des surfaces*," Journal de l'école polytechnique, cahier 39 (1862).

† Annals of Mathematics, l. c.

of revolution these cylinders are similar. Stübler* determined all minimal surfaces which are envelopes of similar cylinders which may be brought into similar position by rotation about the Z-axis, finding that such surfaces are given by three different forms of $F(u)$ in (1), each depending on several parameters, these surfaces including those applicable to surfaces of revolution and spiral minimal surfaces.

9. **Symmetries of $S_{n\alpha}$.** From the equations (8) of $S_{n\alpha}$ it appears that if α and r are replaced by $-\alpha$ and $1/r$ respectively the values of x and y are unchanged and the sign of z is reversed; it follows that $S_{n\alpha}$ and $S_{n-\alpha}$ have symmetry with respect to the xy plane, symmetrical points being given by r, ϕ and $1/r, \phi$. If in (8) n is changed to $-n$, and consequently β to $-\beta$, if then r, ϕ are replaced by $1/r, -\phi$ respectively, x is unchanged, and the signs of both y and z are reversed; then $S_{n\alpha}$ and $S_{-n\alpha}$ are symmetrical with respect to the X axis. It follows that $S_{n\alpha}$ and $S_{-n-\alpha}$ are symmetrical with respect to the xz plane. If the isothermal parameters $\log r, \phi$ are used as coördinates symmetrical points in all three cases are given by changing the signs of one or of both coördinates.

10. **Two properties characteristic of $S_{n\alpha}$.** We prove a characteristic property of spiral minimal surfaces together with a similar characteristic property of minimal surfaces applicable to surfaces of revolution given by the following theorem, a converse of theorems given in §8. If every Minding parallel (r) of a real minimal surface is an isogonal trajectory of either family of lines of curvature or of the level curves of the surface the minimal surface is a spiral surface or a helicoid; if every Minding meridian (ϕ) of a real minimal surface is an isogonal trajectory of either of the families named the minimal surface is applicable to a surface of revolution.

To prove the first part of this theorem we observe that the condition that all curves (r) be isogonal trajectories of either of the families named may be expressed by the equation

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f(uv),$$

where F and Φ are the conjugate functions in (1) and f is unknown. If this equation is differentiated first with respect to u and then with respect to v, and the two values of f' equated, we have

$$\frac{2F(u) + uF'(u)}{F(u)} = -\frac{2\Phi(v) + v\Phi'(v)}{\Phi(v)} = ni,$$

where n is a real constant. These give

$$F(u) = cu^{-2+ni}, \qquad \Phi(v) = \bar{c}v^{-2-ni},$$

*Mathematische Annalen, vol. 75 (1914).

so that the minimal surface is a spiral surface, or, if $n = 0$, a helicoid. The condition that each curve (ϕ) of (1) be an isogonal trajectory of any one of the families named is

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f\left(\frac{u}{v}\right),$$

which leads in a similar way to

$$F(u) = cu^{m-2}, \qquad \Phi(v) = \bar{c}v^{m-2},$$

where m is a real constant. The only real minimal surfaces such that both families of curves (r) and (ϕ) have the isogonal property are the helicoids. It is easily shown that it is only on the minimal helicoids that either of these families, (r) and (ϕ), forms an isogonal *system* with any of the three other families named, for only in this case is the angle of intersection constant.

It appeared in § 4 that for spiral minimal surfaces part of the locus L is a plane containing the center of the Schwarz ellipses;* we have elsewhere proved that the same thing is true for minimal surfaces applicable to surfaces of revolution if m is different from zero or plus or minus one. We now prove that this property is characteristic of these two classes of real minimal surfaces.

Since the equations of any minimal surface in any position are given by (1) we may without restriction suppose the equations of a minimal surface for which part of L is a plane containing the center of the Schwarz ellipses to have the form (1) and this plane part of L to be the xy plane, $z = 0$. From (3) it follows that, for all u, v,

$$z = U_3 \sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_3 \sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}} = 0,$$

for some determination of the first radical. This gives one of the identities

$$\frac{U_3}{\sqrt{\sum U_1^2}} = \frac{V_3}{\sqrt{\sum V_1^2}} = c \qquad \text{or} \qquad \frac{U_3}{\sqrt{\sum U_1^2}} = -\frac{V_3}{\sqrt{\sum V_1^2}} = c,$$

where $\sqrt{\sum U_1^2}$ and $\sqrt{\sum V_1^2}$ are conjugate and c is a constant real in the first case and pure imaginary in the second. We consider the equation

$$U_3 / \sqrt{\sum U_1^2} = c,$$

and write

$$\sqrt{\sum U_1^2} = \rho, \qquad U_1 = \lambda\rho, \qquad U_2 = \mu\rho, \qquad U_3 = c\rho.$$

From the last equations

$$\lambda^2 + \mu^2 + c^2 = 1, \qquad \lambda\lambda' + \mu\mu' = 0,$$

$$U_1' = \lambda\rho' + \lambda'\rho, \qquad U_2' = \mu\rho' + \mu'\rho, \qquad U_3' = c\rho'.$$

*American Journal of Mathematics, l. c.

From the fact that $u+v, i(v-u), uv-1$ are proportional to the direction cosines of the normal to (1) it follows that for all u, v

$$(u+v)U_1' + i(v-u)U_2' + (uv-1)U_3' = 0,$$

which may be replaced by the two equations

$$u(U_1' - iU_2') - U_3' = 0, \qquad U_1' + iU_2' + uU_3' = 0.$$

The last two equations give $\Sigma U_1'^2 = 0$, from which, substituting the values given for U_1', U_2', U_3'

$$\rho'^2 + \rho^2(\lambda'^2 + \mu'^2) = 0, \qquad \rho = Ce^{\pm i\int\sqrt{\lambda'^2+\mu'^2}du}.$$

Substituting the values of U_1', U_2', U_3' in the two linear equations connecting these quantities, then replacing in these ρ' by $\pm \rho i\sqrt{\lambda'^2+\mu'^2}$, and cancelling ρ, which cannot be identically zero unless (1) is a plane,* we find

$$\frac{\lambda' + i\mu'}{\mp\sqrt{\lambda'^2+\mu'^2}} = i(\lambda + \mu i + cu), \qquad \frac{\lambda' - i\mu'}{\mp\sqrt{\lambda'^2+\mu'^2}} = i\left(\lambda - \mu i - \frac{c}{u}\right).$$

Multiplying the last two equations,

$$(\lambda + i\mu + cu)\left(\lambda - i\mu - \frac{c}{u}\right) + 1 = 0.$$

From this and $\lambda^2+\mu^2+c^2=1$ we find

$$\lambda + i\mu = \frac{1 - c^2 \pm \sqrt{1-c^2}\,u}{c}, \qquad \lambda - i\mu = -\frac{1-c^2 \mp \sqrt{1-c^2}}{cu}.$$

Differentiating and multiplying the results, $\lambda'^2+\mu'^2 = (c^2-1)/u^2$, and

$$\rho = Ku^{\pm\sqrt{1-c^2}}, \quad U_3 = c\rho = cKu^{\pm\sqrt{1-c^2}}.$$

If $c=0$ the surface (1) is the plane, $z=0$; if $c=\pm 1$ the surface is the plane $z=$ constant; for all other values of c the function $F(u)$ of (1) is given by

$$F(u) = \frac{1}{u}U_3' = cKu^{-2\pm\sqrt{1-c^2}}.$$

If c is pure imaginary we have $F(u) = Au^{m-2}$, where m is a real constant numerically greater than one; if c is real and numerically less than one, $F(u)$ has the same form and m is real and numerically less than one. Such values of c give all minimal surfaces applicable to surfaces of revolution, except that

* My paper, American Journal of Mathematics, l.c.